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Maximum-Likelihood Estimation of Time-Varying Delay

John A. Stuller
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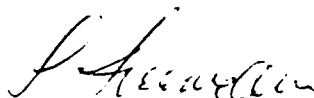
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PREFACE

This report is based upon work performed by the author at the University of Missouri, at Rolla, before he joined NUSC on IPA assignment on 5 August 1985.

The Technical Reviewer for this report was Dr. G.C. Carter (Code 3314).

Reviewed and Approved: 30 July 1986

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Head, Surface Ship Sonar Department

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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	

4. PERFORMING ORGANIZATION REPORT NUMBER(S) TR 7665	5. MONITORING ORGANIZATION REPORT NUMBER(S)

6a. NAME OF PERFORMING ORGANIZATION Naval Underwater Systems Center	6b. OFFICE SYMBOL (If applicable) Code 3314	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) New London Laboratory New London, CT 06320		7b. ADDRESS (City, State, and ZIP Code)

8a. NAME OF FUNDING/SPONSORING ORGANIZATION Naval Underwater Systems Center	8b. OFFICE SYMBOL (If applicable) Code 3314	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER		
8c. ADDRESS (City, State, and ZIP Code) New London Laboratory New London, CT 06320		10. SOURCE OF FUNDING NUMBERS		

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.
	P24410		

11. TITLE (Include Security Classification)

MAXIMUM-LIKELIHOOD ESTIMATION OF TIME-VARYING DELAY

12. PERSONAL AUTHOR(S)
John A. Stuller

13a. TYPE OF REPORT IPA Assignee Report	13b. TIME COVERED FROM 1985 TO 1986	14. DATE OF REPORT (Year, Month, Day) 1986, July 30	15. PAGE COUNT 67

16. SUPPLEMENTARY NOTATION

17	COSATI CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Canonical Realizations Maximum Likelihood Estimation
17	01		Log-Likelihood Function Time-Varying Delay
			Matrix-Impulse Response Uncorrelated Noise

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

This report presents, for the first time, the exact theoretical solution to the problem of maximum-likelihood (ML) estimation of time-varying delay, $d(t)$, between a random signal, $s(t)$, received at one point in the presence of uncorrelated noise and the time-delayed, scaled version $as(t - d(t))$ of that signal received at another point in the presence of uncorrelated noise. The signal is modeled as a sample function of a nonstationary Gaussian random process and the observation interval is arbitrary. This report represents a generalization of that of Knapp and Carter (The Generalized Correlation Method for Estimation of Time Delay, IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. ASSP-24, no. 4, August 1976, pp. 320-327), who derived the ML estimator for the case that the delay is constant, $d(t) = d_0$; the signal process is stationary; and the received processes are observed over the infinite interval, $[-\infty, +\infty]$. We show that the ML estimator of $d(t)$ can be implemented in any of four canonical forms,

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED	
22a. NAME OF RESPONSIBLE INDIVIDUAL John A. Stuller	22b. TELEPHONE (Include Area Code) (203) 440-6469	22c. OFFICE SYMBOL Code 3314

DD FORM 1473, 84 MAR   83 APR edition may be used until exhausted   SECURITY CLASSIFICATION OF THIS PAGE
All other editions are obsolete

19. (Cont'd)

which in general, are time-varying systems. We also show that our results reduce to a generalized crosscorrelator for the special case treated by Knapp and Carter.

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LIST OF SYMBOLS

T_i: Initial observation time.

T_f: Final observation time.

MMSE and LMMSE: Unconstrained minimum mean square error and linear minimum mean square error, respectively. Here these terms can be used interchangeably because the processes are Gaussian.

Subscript n: Noncausal estimate or system.

Subscript c: Causal estimate or system.

ML: Maximum likelihood

T (superscript): Vector or matrix transposition.

$\text{Tr}\{\cdot\}$: Trace of matrix argument.

$|\underline{\theta}|^2$: Squared Euclidean norm, $|\underline{\theta}|^2 = \underline{\theta}^T\underline{\theta}$.

$\delta(t)$: Dirac delta function.

δ_{ij}: Kronecker delta function.

$\text{SGN}(\widetilde{A})$: Algebraic sign function. $\text{SGN}(\widetilde{A}) = 1$ for $\widetilde{A} > 0$,

$\quad\quad \text{SGN}(\widetilde{A}) = -1$ for $\widetilde{A} < 0$.

Random or unknown variables are denoted by lower case letters. Given or assumed values of random or unknown variables are denoted by upper case letters.

MAXIMUM-LIKELIHOOD ESTIMATION

OF TIME-VARYING DELAY

1. INTRODUCTION

In their classic paper, "The Generalized Correlation Method for Estimation of Time Delay,"[1] Knapp and Carter presented the solution to the problem of maximum-likelihood (ML) estimation of constant delay, d_0, between signals received at two spatially separated sensors in the presence of uncorrelated noise. The received waveforms were modeled mathematically as

$$r_a(t) = s(t) + n_a(t), \quad -\infty < t < +\infty , \tag{1-1a}$$

$$r_b(t) = \tilde{a}s(t - d_0) + n_b(t), \quad -\infty < t < +\infty , \tag{1-1b}$$

where $\tilde{a}$ was a relative attenuation constant and $s(t)$, $n_a(t)$, and $n_b(t)$ were uncorrelated, stationary Gaussian random processes. Knapp and Carter[1] showed that the ML estimator of d_0 can be realized by a pair of prefilters followed by a crosscorrelator. Their solution is identical to that proposed by Hannon and Thomson[2,3] whose motivation for estimating delay was to improve estimates of the spectra, cross spectra, and coherence of stationary time series. Here we present a major generalization of these preceding analyses that includes (1) arbitrary time-varying delay, $d(t)$; (2) nonstationary random signal process; and (3) arbitrary observation interval.

Previous attempts to extend the theoretical solution described in references 1, 2, and 3 have been relatively limited in scope. Assuming a

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stationary signal process and an infinite observation interval, Knapp and Carter[4] obtained an approximate ML estimator of both differential delay and Doppler for a source whose relative velocity is much less than the signal propagation velocity. Here the estimator structure included a time-compander following one of the two prefilters before crosscorrelation to compensate for the Doppler time scaling of the waveform. Wax[5] generalized the analysis to include differential phase. Several other authors[6-9] have described the degradation in compensated and uncompensated crosscorrelator outputs due to motion of the source. Beyond these relatively limited theoretical studies, there has been both a need for and a continuing effort to develop practical algorithms that estimate time-varying delay more generally.[10-16] The new and general theory presented here can provide guidance to that effort, as well as fresh insights into previous theoretical results.

We model the problem of time-varying-delay estimation as follows: A vector of real waveforms,

$$\underline{r}(t) = \begin{bmatrix} r_1't) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} s(t) \\ \tilde{a}s(t - d(t)) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} , \qquad (1-2)$$

is observed on the interval $[T_i, T_f]$, where T_i and T_f denote initial and final observation time, respectively. *For convenience, we define* $\underline{r}(t)$ as zero for t outside this interval. The signal s(t) is a sample function of a zero-mean Gaussian random process having covariance function

$$R_s(t_1,t_2) = E\{s(t_1)s(t_2)\} . \qquad (1-3)$$

The delayed and attenuated signal $\tilde{a}s(t - d(t))$ is related to $s(t)$ through a nonrandom but unknown invertible linear operator,

$$L_{d(t),\tilde{a}}\{s(t)\} = \tilde{a}s(t - d(t)) \ . \tag{1-4}$$

The noise waveforms $w_1(t)$ and $w_2(t)$ are sample functions of white Gaussian random processes having covariance functions

$$R_{w_1}(t_1,t_2) = R_{w_2}(t_1,t_2) = \frac{N_0}{2}\delta(t_1 - t_2) \ . \tag{1-5}$$

The signal process and noise processes are mutually independent. The attenuation factor $\tilde{a}$ and delay function $d(t)$ in (1-2) and (1-4) are nonrandom but unknown. Since $d(t)$ represents delay, we will assume here that $d(t) \geq 0$. This restriction can be removed with a somewhat more lengthy analysis. The attenuation constant $\tilde{a}$ can be any nonzero real number. The problem is to estimate $d(t)$ and $\tilde{a}$.

The model in (1-2) through (1-5) assumes white noise processes and a nondispersive (frequency independent) propagation medium for simplicity. One can extend the developments of the following sections to include nonwhite noise by applying noise whitening techniques similar to those described in reference 17, p. 290. One can also include dispersion, as well as time varying delay, by replacing the operation (1-4) by a more general invertible time-varying linear system. Hamon and Hannan[18] have previously described an approximate ML solution to the time-delay-estimation problem for dispersive systems.

Although the details of our derivation differ substantially from those in reference 1, the estimation criterion is identical. We seek a function, the log-likelihood function $\ln\Lambda(D(t),\widetilde{A})$, whose value is an indicator of the likelihood that a hypothetical delay function, $D(t)$, and attenuation constant, $\widetilde{A}$, caused a particular received vector waveform, $R(t)$, $T_i \leq t \leq T_f$. The function $D(t)]_{ML}$ and the constant $\widehat{\widetilde{A}}]_{ML}$ jointly maximizing $\ln\Lambda(D(t),\widetilde{A})$ are the ML estimates of $d(t)$ and $\widetilde{a}$, respectively, when $\underline{R}(t)$ is the received vector waveform.

This report is organized as follows: The log-likelihood function $\ln\Lambda(D(t),\widetilde{A})$, derived in section 2, is shown to depend upon the minimum mean square error (MMSE) estimators of $s(t)$ and $\widetilde{a}s(t - d(t))$ from $\underline{r}(t)$ conditioned on given attenuation and delay. We show how to implement these estimators in section 3. In section 4 we obtain the four alternative systems for computing $\ln\Lambda(D(t),\widetilde{A})$. In section 5 we show that the general solution to the problem of ML estimation of $d(t)$ reduces to the generalized crosscorrelator receiver in reference 1 for the special case that $d(t)$ is a constant, the signal process is stationary, and the observation interval is long.

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2. DERIVATION OF THE LOG-LIKELIHOOD FUNCTION

The derivation of the log-likelihood function, $\ln\Lambda(D(t),\widetilde{A})$, is somewhat lengthy and, therefore, has been separated into three parts. The first part of the derivation, described in subsection 2.1, obtains a series form for $\ln\Lambda(D(t),\widetilde{A})$ using the generalized Karhunen-Loeve expansion. The result is given in equations (2-16), (2-17), and (2-18). The second and third parts, in subsections 2.2 and 2.3, show that the series (2-17) and (2-18) can be put into the closed forms (2-19) and (2-36), respectively, that, in turn, depend upon the noncausal and the causal MMSE estimators of $s(t)$ and $\hat{a}s(t - d(t))$ from $\underline{r}(t)$ conditioned on given attenuation and delay.

The developments in subsections 2.1, 2.2, and 2.3 are basically extensions of the material in references 17 (pp. 203-205 and 221-223) and 19 (pp. 22, pp. 170-173) from scalar to vector random processes.

2.1 SERIES FORM

The problem of estimating attenuation and time-varying delay can be reframed as a parameter-estimation problem. One way to do this is to represent the time-varying delay, $d(t)$, by a generalized Fourier series,

$$d(t) = \sum_{i=1}^{\infty} d_i \psi_i(t) \; ; \; T_i \le t \le T_f , \tag{2-1}$$

where

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$$d_i = \int_{T_i}^{T_f} d(t)\psi_i(t)dt \tag{2-2}$$

and where $\{\psi_i(t)\}$ is any convenient set of basis functions that is complete and orthonormal (CON) over the interval $[T_i, T_f]$. Because $d(t)$ is nonrandom but unknown, the coefficients d_1, d_2, d_3, ... are nonrandom but unknown. The substitution of (2-1) into $s(t - d(t))$ yields a function that depends upon the basis set $\{\psi_i(t)\}$, the vector of unknown coefficients $\underline{d} = (d_1, d_2, ...)$, and t. To show the dependency on $\underline{d}$ explicitly, we will denote this function by $s(t;\underline{d})$; that is,

$$s(t;\underline{d}) \triangleq s(t - \sum_{i=1}^{\infty} d_i\psi_i(t)). \tag{2-3}$$

It follows from notation (2-3) that

$$s(t;\underline{0}) = s(t) \quad . \tag{2-4}$$

We now write $\underline{r}(t)$ of (1-2) as

$$\underline{r}(t) = \underline{s}(t;\underline{d},\tilde{a}) + \underline{w}(t) , \tag{2-5}$$

where

$$\underline{r}(t) = (r_1(t) \ r_2(t))^T , \tag{2-6a}$$

$$\underline{s}(t;\underline{d},\tilde{a}) \triangleq (s(t;\underline{0}) \ \tilde{a}s(t;\underline{d}))^T , \tag{2-6b}$$

and

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$$\underline{w}(t) = (w_1(t) \; w_2(t))^T \; . \tag{2-6c}$$

The problem of estimating the unknown delay d(t) and relative attenuation constant $\widetilde{a}$ from $\underline{r}(t)$ in (1-2) is equivalent to that of estimating the unknown vector $\underline{d}$ and scalar $\widetilde{a}$ from $\underline{r}(t)$ in (2-6). We have considered the generalized Fourier series representation of d(t) in (2-1) because it is both well-known and general. Other techniques for representing d(t) as a vector may be preferred in a particular application. For example, if it is known a priori that $d(t) = a_0 + a_1 t + a_2 t^2$ for $T_i \leq t \leq T_f$ (where a_0, a_1, and a_2 are unknown), one can set $\underline{d} = (a_0, a_1, a_2)^T$ to obtain a four-parameter-estimation problem involving the unknown attenuation $\widetilde{a}$ and the physically meaningful constants a_0, a_1, and a_2.

Notice that if d(t) is a known function, D(t), and if $\widetilde{a}$ is a known constant $\widetilde{A}$, then $\widetilde{a}s(t - d(t))$ is related to s(t) simply by a known linear transformation, $L_{D(t),\widetilde{A}}\{s(t)\} = \widetilde{A}s(t - D(t))$. Therefore, for given $\widetilde{a}$ and d(t), the signal s(t) and its delayed, attenuated version $\widetilde{a}s(t - d(t))$ are jointly Gaussian random processes. Let $\underline{D}$ be the vector of coefficients corresponding D(t). Then, for $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$, $\underline{r}(t)$ in (2-6) is a Gaussian random vector process having mean zero and 2 X 2 matrix covariance function,

$$K_{\underline{r};\underline{d},\widetilde{a}}(t,u,;\underline{D},\widetilde{A}) \triangleq E\{\underline{r}(t)\underline{r}^T(u)|\underline{d} = \underline{D}, \widetilde{a} = \widetilde{A}\}$$

$$= E\{\underline{s}(t;\underline{D},\widetilde{A})\underline{s}^T(u;\underline{D},\widetilde{A})\} + E\{\underline{w}(t)\underline{w}^T(u)\}$$

$$= K_{\underline{s};\underline{d},\widetilde{a}}(t,u;\underline{D},\widetilde{A}) + \frac{N_0}{2} I \, \delta(t - u) \; , \tag{2-7}$$

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where I is the 2 X 2 identity matrix.

We proceed by representing vector process $\underline{r}(t)$ as an infinite dimensional vector $\underline{r}$ using the generalized Karhunen-Loeve expansion (references 17 (pp. 221-223) and 20). We define

$$\underline{r}_N(t) \triangleq \sum_{i=1}^{N} r_i \underline{\phi}_i(t;\underline{D},\tilde{A}) , \qquad (2-8)$$

where

$$r_i = \int_{T_i}^{T_f} \underline{\phi}_i^T(t;\underline{D},\tilde{A})\underline{r}(t)dt ; \quad i = 1, 2, \ldots, N \qquad (2-9)$$

and where the $\underline{\phi}_i(t;\underline{D},\tilde{A})$ are the normalized vector eigenfunctions of the matrix covariance function $K_{\underline{s};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A})$. We assume that $\{\underline{\phi}_i(t;\underline{D},A)\}$ is complete. The normalized vector eigenfunctions are two-element-column vectors that satisfy the equations

$$\lambda_i(\underline{D},\tilde{A})\underline{\phi}_i(t;\underline{D},\tilde{A}) = \int_{T_i}^{T_f} K_{\underline{s};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A})\underline{\phi}_i(u;\underline{D},\tilde{A}) \, du ; \quad T_i \leq t \leq T_f \qquad (2-10)$$

and

$$\int_{T_i}^{T_f} \underline{\phi}_i^T(t;\underline{D},\tilde{A})\underline{\phi}_j(t;\underline{D},\tilde{A}) \, dt = \delta_{ij} , \qquad (2-11)$$

where $\lambda_i(\underline{D},\tilde{A})$ is the (scalar) eigenvalue associated with $\underline{\phi}_i(t;\underline{D},\tilde{A})$. With the $\underline{\phi}_i(t;\underline{D},\tilde{A})$ so specified, it follows that

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$$\underline{r}(t) = \underset{N \to \infty}{l.i.m.} \ \underline{r}_N(t) \ . \tag{2-12}$$

This is the generalized Karhunen-Loeve expansion of $\underline{r}(t)$.

Because $\{\underline{\phi}_i(t;\underline{D},A)\}$ is complete, one can represent $\underline{r}(t)$ by (2-12) (using (2-8) and (2-9)) for hypothetical or assumed values of $\underline{D}$ and $\widetilde{A}$. It is easy to show that if the assumed values of $\underline{D}$ and $\widetilde{A}$ are the true values of the unknown quantities $\underline{d}$ and $\widetilde{a}$, respectively, then the r_i's in (2-9) are statistically independent Gaussian random variables having zero means and variances,

$$E\left\{r_i^2 | \underline{d} = \underline{D}, \ \widetilde{a} = \widetilde{A}\right\} = \lambda_i(\underline{D},\widetilde{A}) + \frac{N_o}{2} \ ; \ i = 1, \ 2, \ \dots, \ N \ . \tag{2-13}$$

The joint probability density function of the r_i conditioned on $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$ is, therefore,

$$p_{\underline{r}_N;\underline{d},\widetilde{a}}(\underline{R}_N;\underline{D},\widetilde{A}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi[\lambda_i(\underline{D},\widetilde{A}) + \frac{N_o}{2}]}} \exp\left\{ - \frac{R_i^2}{2[\lambda_i(\underline{D},\widetilde{A}) + \frac{N_o}{2}]} \right\},$$

$$\tag{2-14}$$

where $\underline{r}_N = (r_1 \ r_2 \ \dots \ r_N)^T$ and $\underline{R}_N = (R_1 \ R_2 \ \dots R_N)^T$. To obtain the likelihood function associated with $\underline{r}(t)$, we take the logarithm and the limit $N \to \infty$. This leads to a convergence difficulty that can be bypassed in the usual way (see reference 17, p. 274) of dividing (2-14) by the function

$$f(\underline{R}_N) = \prod_{i=1}^{N} \frac{1}{\sqrt{\pi N_o}} \; \exp\left\{ - \frac{R_i^2}{N_o} \right\} \; . \tag{2-15}$$

The result of this division is still a likelihood function because $f(\underline{R}_N)$ does not depend upon $\underline{D}$ or $\widetilde{A}$. After dividing (2-14) by (2-15), we take the logarithm and the limit $N \to \infty$. The result is the log-likelihood function

$$\ln\Lambda(\underline{D},\widetilde{A}) = \ell_R(\underline{D},\widetilde{A}) + \ell_B(\underline{D},\widetilde{A}) \; . \tag{2-16}$$

where

$$\ell_R(\underline{D},\widetilde{A}) \triangleq \frac{1}{N_o} \sum_{i=1}^{\infty} \frac{\lambda_i(\underline{D},\widetilde{A})}{\lambda_i(\underline{D},\widetilde{A}) + N_o/2} \; R_i^2 \tag{2-17}$$

and

$$\ell_B(\underline{D},\widetilde{A}) \triangleq - \frac{1}{2} \sum_{i=1}^{\infty} \ln[1 + \frac{2}{N_o} \lambda_i(\underline{D},\widetilde{A})] \quad . \tag{2-18}$$

2.2   CLOSED FORM FOR DATA DEPENDENT TERM $\ell_R(\underline{D},\widetilde{A})$

The first term in (2-16), $\ell_R(\underline{D},\widetilde{A})$, can be written as

$$\ell_R(\underline{D},\widetilde{A}) = \frac{1}{N_o} \int_{T_i}^{T_f} \int_{T_i}^{T_f} \underline{R}^T(t)\underline{H}_n(t,v;\underline{D},\widetilde{A})\underline{R}(v)dtdv \; , \tag{2-19}$$

where

$$\underline{H}_n(t,v;\underline{D},\widetilde{A}) \triangleq \sum_{i=1}^{\infty} \frac{\lambda_i(\underline{D},\widetilde{A})}{\lambda_i(\underline{D},\widetilde{A}) + N_0/2} \underline{\emptyset}_i(t;\underline{D},\widetilde{A})\underline{\emptyset}_i^T(v;\underline{D},\widetilde{A}) \; ;$$

$$T_i \le t,v \le T_f \; , \tag{2-20}$$

and $\underline{R}(t)$ is the sample vector function of $\underline{r}(t)$ corresponding to the vector $\underline{R}$:

$$\underline{R}(t) = \sum_{i=1}^{\infty} R_i\underline{\emptyset}_i(t;\underline{D},\widetilde{A}) \; , \tag{2-21}$$

with

$$R_i = \int_{T_i}^{T_f} \underline{\emptyset}_i^T(t;\underline{D},\widetilde{A})\underline{R}(t)dt \; . \tag{2-22}$$

Equation (2-19) can be verified by substituting $\underline{H}_n(t,v;\underline{D},\widetilde{A})$ in (2.20) into (2-19) and using (2-22). An interpretation of (2-19) is obtained by considering the following noncausal linear estimate of $\underline{s}(t;\underline{D},\widetilde{A})$:

$$\hat{\underline{s}}_n(t;\underline{D},\widetilde{A}) = \int_{T_i}^{T_f} \underline{H}_n(t,v;\underline{D},\widetilde{A})\underline{r}(v)dv, \; T_i \le t \le T_f \; , \tag{2-23}$$

where the subscript "n" is used to denote a noncausal estimate or system.

It follows from (2-23) and (2-7) that

$$E\left\{[\underline{s}(t;\underline{d},\tilde{a}) - \hat{\underline{s}}_n(t;\underline{D},\tilde{A})]\underline{r}^T(u)|\underline{d} = \underline{D},\tilde{a} = \tilde{A}\right\} = K_{\underline{s};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A})$$

$$- \frac{N_0}{2} \underline{H}_n(t,u;\underline{D},\tilde{A}) - \int_{T_i}^{T_f} \underline{H}_n(t,v;\underline{D},\tilde{A})K_{\underline{s};\underline{d},\tilde{a}}(v,u;\underline{D},\tilde{A})dv \; ; \; T_i \leq t,u \leq T_f.$$

$$(2\text{-}24)$$

According to the matrix version of Mercer's Theorem:[20]

$$K_{\underline{s};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A}) = \sum_{i=1}^{\infty} \lambda_i(\underline{D},\tilde{A})\underline{\phi}_i(t;\underline{D},\tilde{A})\underline{\phi}_i^T(u;\underline{D},\tilde{A}) \; ; \; T_i \leq t, \, u \leq T_f \; .$$

$$(2\text{-}25)$$

By substituting (2-25) and (2-20) into the right-hand side of (2-24) and using (2-11) one obtains

$$K_{\underline{s};\underline{d},\tilde{a}}(t,u;\underline{D},\tilde{A}) - \frac{N_0}{2} \underline{H}_n(t,u;\underline{D},\tilde{A}) - \int_{T_i}^{T_f} \underline{H}_n(t,v;\underline{D},\tilde{A})K_{\underline{s};\underline{d},\tilde{a}}(v,u;\underline{D},\tilde{A})dv$$

$$= 0 \; ; \; T_i \leq t_i, \, u \leq T_f \; , \qquad\qquad (2\text{-}26a)$$

which yields

$$E\left\{[\underline{s}(t;\underline{d},\tilde{a}) - \hat{\underline{s}}_n(t;\underline{D},\tilde{A})]\underline{r}^T(u) \mid \underline{d} = \underline{D},\tilde{a} = \tilde{A}\right\} = \underline{0} \; ; \; T_i \leq t,u \leq T_f. \qquad (2\text{-}26b)$$

Therefore, if $\underline{D}$ and $\tilde{A}$ are the true values of $\underline{d}$ and $\tilde{a}$, then the estimation error

$$\underline{e}_n(t;\underline{d},\tilde{a},\underline{D},\tilde{A}) \triangleq \underline{s}(t;\underline{d},\tilde{a}) - \hat{\underline{s}}_n(t;\underline{D},\tilde{A}) \qquad\qquad (2\text{-}27)$$

is **orthogonal** to $\underline{r}(u)$, $T_i \leq u \leq T_f$. Consequently (see reference 21,

p. 390) $\hat{\underline{s}}_n(t;\underline{D},\widetilde{A})$ in (2-23) is the noncausal point LMMSE estimate of

$\underline{s}(t;\underline{d},\widetilde{a})$ from $\underline{r}(u)$, $T_i \leq u \leq T_f$, given that $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$ and

$\underline{H}_n(t,v;\underline{D},\widetilde{A})$ is the impulse response of the noncausal point LMMSE

estimator. Note that the 2-by-2 matrix function $\underline{H}_n(t,v;\underline{D},\widetilde{A})$ is the

solution to equation (2-26a). As will be described in section 4, the

substitution of (2-23) into (2-19) results in a vector estimator-correlator

realization for $\ell_R(\underline{D},\widetilde{A})$.

The error convariance matrix of the noncausal point LMMSE estimate of

$\underline{s}(t;\underline{d},\widetilde{a})$ conditioned on $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$ is

$$\underline{E}_n(t;\underline{D},\widetilde{A}) \triangleq E\left\{\underline{e}_n(t;\underline{d},\widetilde{a},\underline{D},\widetilde{A})\underline{e}_n^T(t;\underline{d},\widetilde{a},\underline{D},\widetilde{A})|\underline{d} = \underline{D}, \widetilde{a} = \widetilde{A}\right\} . \qquad (2\text{-}28)$$

By substituting (2-27) into (2-28) and using (2-23) and (2-26), one obtains

$$\underline{E}_n(t;\underline{D},\widetilde{A}) = \frac{N_o}{2} \underline{H}_n(t,t;\underline{D},\widetilde{A}) . \qquad (2\text{-}29)$$

2.3 CLOSED FORM FOR BIAS TERM $\ell_B(\underline{D},\widetilde{A})$

The term $\ell_B(\underline{D},\widetilde{A})$ in (2-18) can be written in closed form by noting

that, according to (2-10) and (2-11), the eigenvalues $\lambda_i(\underline{D},\widetilde{A})$ and the

vector eigenfunctions $\underline{\phi}_i(t;\underline{D},\widetilde{A})$ depend upon the final observation time

T_f. To indicate this dependency, we write

$$\lambda_i(\underline{D},\widetilde{A}) = \lambda_i(\underline{D},\widetilde{A},T_f) \qquad (2\text{-}30)$$

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$$\underline{\phi}_i(t;\underline{D},\widetilde{A}) = \underline{\phi}_i(t;\underline{D},\widetilde{A},T_f) \ . \tag{2-31}$$

It follows that $\ell_B(\underline{D},\widetilde{A})$ in (2.18) can be rewritten as

$$\ell_B(\underline{D},\widetilde{A}) = -\frac{1}{2}\int_{T_i}^{T_f} dt \ \frac{d}{dt} \ \sum_{i=1}^{\infty} \ln[1 + \frac{2}{N_0} \lambda_i(\underline{D},\widetilde{A},t)]$$

$$= -\frac{1}{N_0}\int_{T_i}^{T_f} dt \ \sum_{i=1}^{\infty}\frac{[d\lambda_i(\underline{D},\widetilde{A},t)]/dt}{1 + (2/N_0)\lambda_i(\underline{D},\widetilde{A},t)} \ , \tag{2-32}$$

where $\lambda_i(\underline{D},\widetilde{A},T_i) = 0$. It can be shown by a straightforward extension of the derivation in reference 17, pp. 204-205, that

$$\frac{d\lambda_i(\underline{D},\widetilde{A},t)}{dt} = \lambda_i(\underline{D},\widetilde{A},t)Tr\left\{\underline{\phi}_i(t;\underline{D},\widetilde{A},t)\underline{\phi}_i{}^T(t;\underline{D},\widetilde{A},t)\right\} \ . \tag{2-33}$$

When we use (2-33) and the fact that $Tr\{\bullet\}$ is a linear operator, (2-32) becomes

$$\ell_B(\underline{D},\widetilde{A}) = -\frac{1}{2}\int_{T_i}^{T_f} Tr\left\{\sum_{i=1}^{\infty}\frac{\lambda_i(\underline{D},\widetilde{A},t)}{\lambda_i(\underline{D},\widetilde{A},t) + N_0/2} \ \underline{\phi}_i(t;\underline{D},\widetilde{A},t)\underline{\phi}_i{}^T(t;\underline{D},\widetilde{A},t)\right\}dt \ .$$

$$\tag{2-34}$$

A closed form for the quantity in the braces in (2-34) is recognized by rewriting (2-20) with the notation of (2-30) and (2-31):

$$\underline{H}_n(t,v;\underline{D},\widetilde{A},T_f) = \sum_{i=1}^{\infty} \frac{\lambda_i(\underline{D},\widetilde{A},T_f)}{\lambda_i(\underline{D},\widetilde{A},T_f) + N_0/2} \, \underline{\phi}_i(t;\underline{D},\widetilde{A},T_f)\underline{\phi}_i^T(v;\underline{D},\widetilde{A},T_f) \ .$$

$$(2\text{-}35)$$

The quantity in the braces in (2-34) is $\underline{H}_n(t,t;\underline{D},\widetilde{A},t)$. Since $\underline{H}_n(t,v;\underline{D},\widetilde{A},t)$ is the matrix impulse response of the point LMMSE estimator of $\underline{s}(t;\underline{d},\widetilde{a})$ from $\underline{r}(v)$, $T_i \leq v \leq t$, given $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$, then $\underline{H}_n(t,v;\underline{D},\widetilde{A},t)$ is, by definition, the matrix impulse response of the **causal** point LMMSE estimator of $\underline{s}(t;\underline{d},\widetilde{a})$ from $\underline{r}(v)$, given $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$. If we denote the causal matrix impulse response by $\underline{H}_c(t,v;\underline{D},\widetilde{A})$, then (2-34) becomes

$$\ell_B(\underline{D},\widetilde{A}) = -\frac{1}{2}\int_{T_i}^{T_f} \mathrm{Tr}[\underline{H}_c(t,t;\underline{D},\widetilde{A})] \ dt \ . \tag{2-36}$$

All the previous equations describing **noncausal** estimation of $\underline{s}(t;\underline{d},\widetilde{a})$ from $\underline{r}(v)$ describe **causal** estimation of $\underline{s}(t;\underline{d},\widetilde{a})$ from $\underline{r}(v)$ when t is substituted for T_f. In particular, with $T_f = t$, equation (2-29) describes the error covariance matrix of the causal LMMSE estimate of $\underline{s}(t;\underline{d},\widetilde{a})$, given $\underline{d} = D$ and $\widetilde{a} = \widetilde{A}$:

$$\underline{E}_c(t;\underline{D},\widetilde{A}) = \frac{N_0}{2} \, \underline{H}_c(t,t;\underline{D},\widetilde{A}) \ . \tag{2-37}$$

The substitution of (2-37) into (2-36) yields an alternative expression for $\ell_B(\underline{D},\widetilde{A})$:

$$\ell_B(\underline{D},\widetilde{A}) = -\frac{1}{N_0}\int_{T_i}^{T_f} \mathrm{Tr}[\underline{E}_c(t;\underline{D},\widetilde{A})] \ dt \ . \tag{2-38}$$

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3. THE MATRIX IMPULSE RESPONSE $\underline{H}_n(t,v;\underline{D},\widetilde{A})$

In this section we derive a simple explicit form for the matrix impulse response $\underline{H}_n(t,v;\underline{D},\widetilde{A})$. It is relatively difficult to obtain this form by solving equation (2-26a). The constructive approach of subsection 3.2 has the advantage of being both mathematically and conceptually simple. Before proceeding with the constructive solution, it will be helpful to derive the explicit form for the inverse of the operator (1-4).

3.1 INVERSE OF OPERATOR (1-4)

By definition, the inverse operator satisfies

$$L^{-1}_{\underline{D},\widetilde{A}}\left\{s(t;\underline{D},\widetilde{A})\right\} = L^{-1}_{D(t),\widetilde{A}}\left\{\widetilde{A}s(t - D(t))\right\} = s(t) . \qquad (3-1)$$

Let $v(t)$ be an arbitrary waveform and try an inverse having the form

$$L^{-1}_{D(t),\widetilde{A}}\left\{v(t)\right\} = \frac{1}{\widetilde{A}} v(\beta(t)) , \qquad (3-2)$$

where $\beta(t)$ is to be determined. Define

$$f(t) = t - D(t) \qquad (3-3)$$

so that by the definition (3-1)

$$L_{D(t),\widetilde{A}}^{-1}\left\{v(f(t))\right\} = \frac{1}{\widetilde{A}}\,v(t) \qquad (3\text{-}4a)$$

Replacing t by f(t) in (3-2) gives

$$L_{D(t),\widetilde{A}}^{-1}\left\{v(f(t))\right\} = \frac{1}{\widetilde{A}}\,v(\beta(f(t)))\;, \qquad (3\text{-}4b)$$

which, when compared with (3-4a), yields

$$\beta(f(t)) = t\;. \qquad (3\text{-}5)$$

Therefore, the inverse operator is given by (3-2), where $\beta(\cdot)$ is the inverse of the function f(t). Since $L_{D(t),\widetilde{A}}\{\cdot\}$ is invertible, the function f(t) is one-to-one. We now proceed to the constructive derivation of $\underline{H}_n(t,v;\underline{D},\widetilde{A})$.

3.2 CONSTRUCTIVE DERIVATION OF $\underline{H}_n(t,v;\underline{D},\widetilde{A})$

The first step in the derivation of $\underline{H}_n(t,v;\underline{D},\widetilde{A})$ is to (noncausally) transform $\underline{r}(t)$, $T_i \le t \le T_f$, into the vector process $\underline{r}'(u)$, $f(T_i) \le u \le T_f$, where

$$\underline{r}'(u) = \begin{bmatrix} \widetilde{A}^{-1} r_2(\beta(u)) \\ 0 \end{bmatrix}; \quad f(T_i) \le u \le T_i\;, \qquad (3\text{-}6a)$$

$$\underline{r}'(u) = \frac{1}{2}\begin{bmatrix} r_1(u) + \widetilde{A}^{-1} r_2(\beta(u)) \\ r_1(u) - \widetilde{A}^{-1} r_2(\beta(u)) \end{bmatrix}; \quad T_i \le u \le f(T_f)\;, \qquad (3\text{-}6b)$$

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$$\underline{r}'(u) = \begin{bmatrix} r_1(u) \\ 0 \end{bmatrix}; \ f(T_f) \leq u \leq T_f \ , \tag{3-6c}$$

and $f(u)$ is defined in (3-3). In (3-6), $\widetilde{A}$ can be regarded as an assumed value for the unknown relative attenuation constant $\widetilde{a}$, and $D(t)$ as an assumed function for the unknown delay function $d(t)$. (We naturally require $D(t) \geq 0$, which implies that $f(t) \leq t$.) Notice that (3-6b) assumes that $f(T_f)$ is not less than T_i. This is equivalent to the assumption that the signal delay does not exceed the observation interval. Since this assumption is likely to be met in most applications, we will retain it in the following. It is not hard to generalize our results to include the case of $f(T_f) < T_i$.

The transformation $\underline{r}(t) \rightarrow \underline{r}'(u)$ is illustrated in figure 3-1, where, for simplicity in interpretation, the noise processes $w_1(t)$ and $w_2(t)$ have been drawn as small ripples. A system block diagram for the transformation is shown in figure 3-2. An examination of equation (3-6), figure 3-1, or figure 3-2 will reveal that the transformation from $\underline{r}(t)$ to $\underline{r}'(u)$ is linear and invertible. Thus, $\underline{r}(t)$, $T_i \leq t \leq T_f$, can be recovered from $\underline{r}'(u)$, $f(T_i) \leq u \leq T_f$, using a linear transformation. It follows from the reversibility theorem (reference 17, p. 289) that the noncausal LMMSE estimate $\hat{\underline{s}}_n(t; \underline{D}, \widetilde{A})$ in equation (2-23), given $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$, can be obtained from $\underline{r}'(u)$. Before describing the structure of the LMMSE estimator, it will be helpful to observe that if $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$, then, from equations (2-5), (2-6), and (3-6),

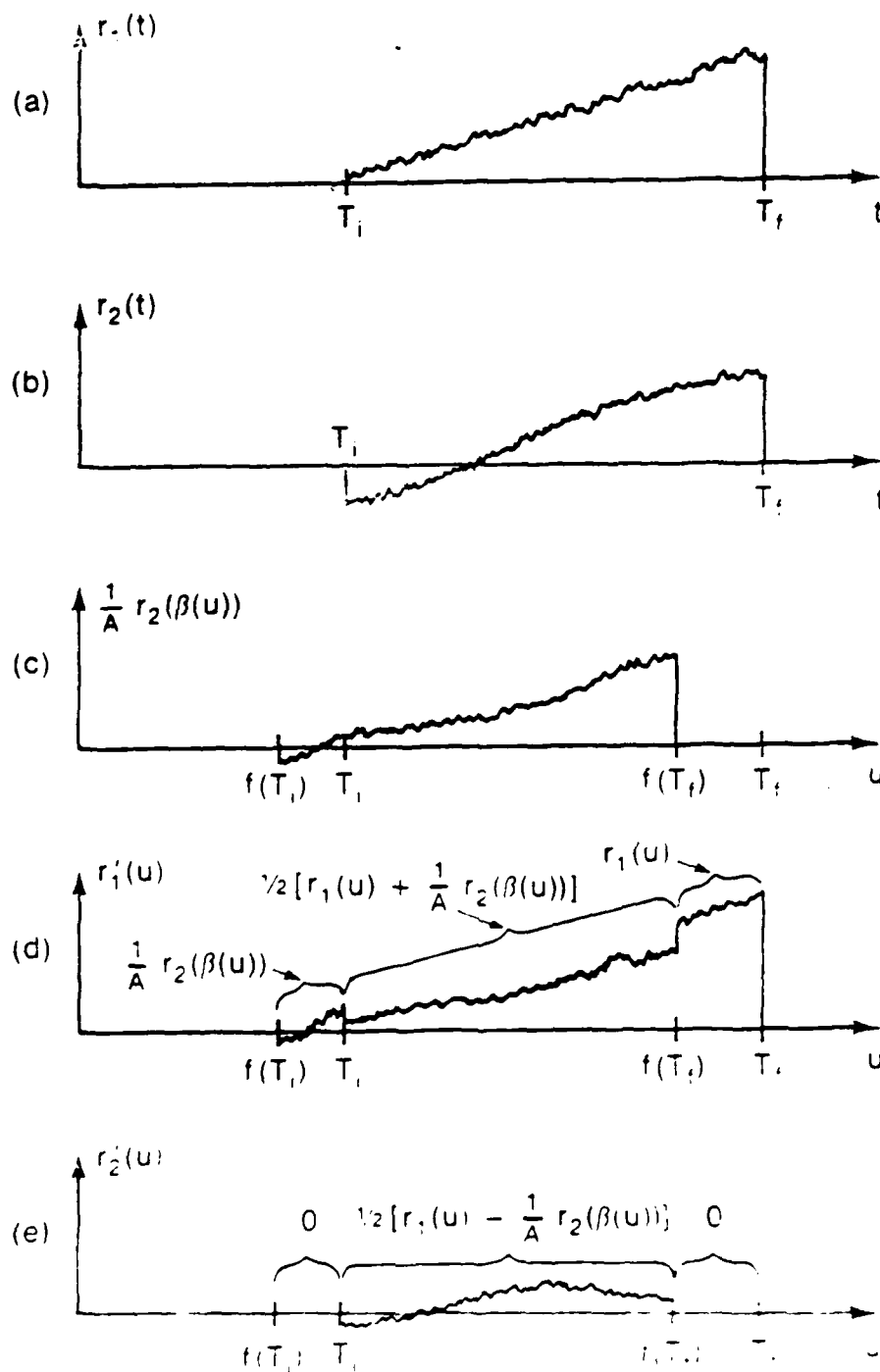Figure 3-1. Components of $\underline{r}(t) = (r_1(t)\ r_2(t))^T$ and $\underline{r}'(t) = (r_1'(t)\ r_2'(t))^T$: (a) $r_1(t)$, (b) $r_2(t)$, (c) Output of Inverse Operator $L_{\underline{D},\widetilde{A}}^{-1}$ in Figure 3-2, (d) $r_1'(u)$, (e) $r_2'(u)$
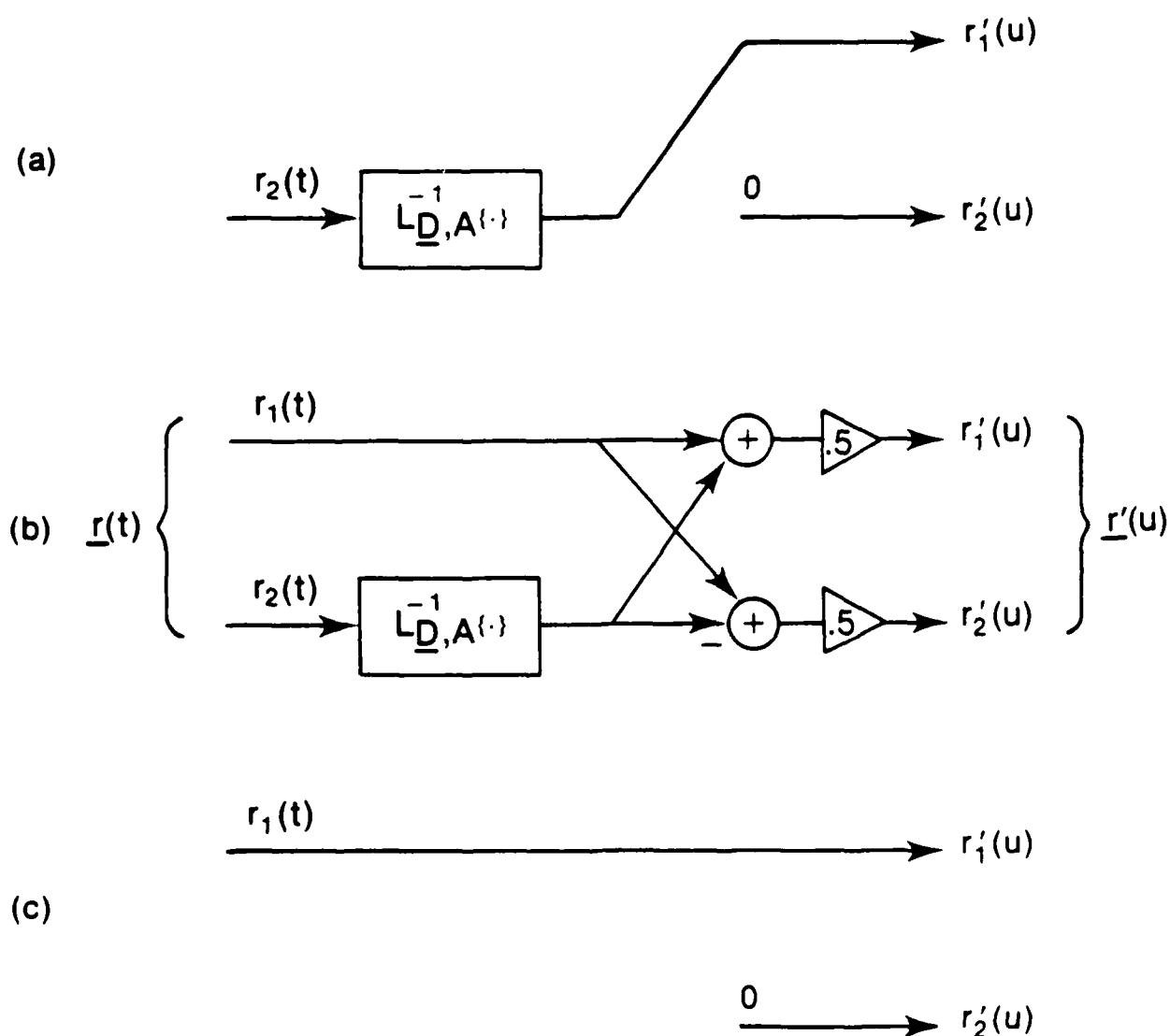
Figure 3-2. System Block Diagram Corresponding to Transformation (3-6):
(a) $f(T_i) \leq u \leq T_i$; (b) $T_i < u \leq f(T_f)$, (c) $f(T_f) < u \leq T_f$

$$\underline{r}'(u) = \underline{0} \; ; \; u < f(T_i),\tag{3-7a}$$

$$\underline{r}'(u) = \begin{bmatrix} s(u) \\ 0 \end{bmatrix} + \begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} \; ; \; f(T_i) \le u \le T_f .\tag{3-7b}$$

and

$$\underline{r}'(u) = \underline{0} \; ; \; T_f < u ,\tag{3-7c}$$

where

$$\begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} = \begin{bmatrix} \widetilde{A}^{-1} w_2(\beta(u)) \\ 0 \end{bmatrix} \; ; \; f(T_i) \le u \le T_i,\tag{3-8a}$$

$$\begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} w_1(u) + \widetilde{A}^{-1} w_2(\beta(u)) \\ w_1(u) - \widetilde{A}^{-1} w_2(\beta(u)) \end{bmatrix} \; ; \; T_i < u \le f(T_f),\tag{3-8b}$$

and

$$\begin{bmatrix} n_1(u) \\ n_2(u) \end{bmatrix} = \begin{bmatrix} w_1(u) \\ 0 \end{bmatrix} \; ; \; f(T_f) < u \le T_f .\tag{3-8c}$$

We present the form of the LMMSE estimator of $s(t)$, $f(T_i) \le t \le T_f$, in the following theorem:

Theorem. The noncausal point LMMSE estimator of $s(t)$ from $\underline{r}'(u)$, $f(T_i) \le t$, $u \le T_f$, conditioned on $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$, is given by the system in figure 3-3, where $f(t,u;\underline{D},\widetilde{A})$ is the impulse response of the noncausal point LMMSE estimator $\hat{n}_1(t)$ of $n_1(t)$ from $n_2(u)$,

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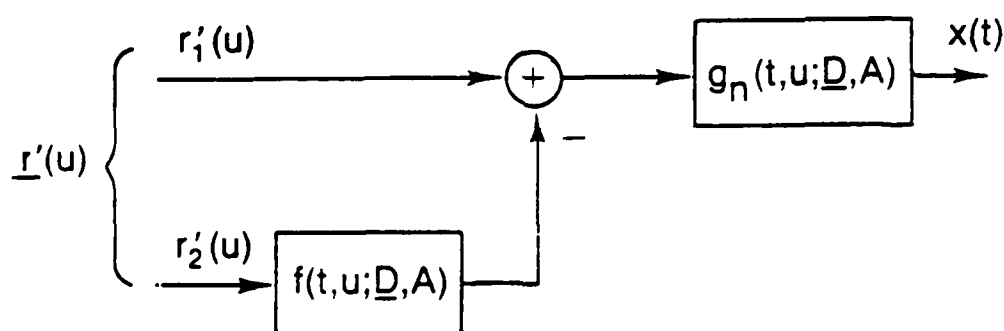
Figure 3-3. Structure of the Noncausal Conditional LMMSE Estimator of
$s(t)$ From $\underline{r}'(u)$, $f(T_i) \leq u \leq T_f$ (When $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, then $x(t)$
equals the noncausal LMMSE estimate of $s(t)$. The impulse
responses $f(t,u;\underline{D},\tilde{A})$ and $g_n(t,u;\underline{D},\tilde{A})$ are defined by the
theorem in section 3. $f(t,u;\underline{D},\tilde{A})$ is specified by
equations (3-30) and (3-31), and $g_n(t,u;\underline{D},\tilde{A})$ is
specified by equations (3-44) and (3-41).

$$\hat{n}_1(t) = \int_{f(T_i)}^{T_f} f(t,u;\underline{D},\widetilde{A})n_2(u)du \; ; \; f(T_i) \le t \le T_f \; , \qquad (3-9)$$

and $g_n(t,u;\underline{D},\widetilde{A})$ is the impulse response of the noncausal point LMMSE estimator $\hat{s}_n(t)$ of $s(t)$ from $s(u) + n_1(u) - \hat{n}_1(u)$,

$$\hat{s}_n(t) = \int_{f(T_i)}^{T_f} g_n(t,u;\underline{D},\widetilde{A}) \; [s(u) + n_1(u) - \hat{n}_1(u)]du \; ; \; f(T_i) \le t \le T_f \; . \qquad (3-10)$$

Proof. According to the **orthogonality principle** (reference 21, p. 390) a linear functional $p = L[\underline{q}]$ is the LMMSE estimate of a random variable p from data vector $\underline{q}(\xi)$ $\xi \epsilon D$ (where D is the domain of the data) if and only if the estimation error $p - \hat{p}$ is orthogonal to $\underline{q}$ for all $\xi \epsilon D$,

$$E\left\{(p - \hat{p})\underline{q}(\xi)\right\} = \underline{0} \; ; \; \xi \epsilon D \; . \qquad (3-11)$$

Therefore, a necessary and sufficient condition that $\hat{s}_n(t)$ be the LMMSE estimate of $s(t)$ from $\underline{r}'(u)$, given that $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$, is that the vector

$$\underline{v}(t,u) \overset{\Delta}{=} E\left\{[s(t) - \hat{s}_n(t)] \; \underline{r}'(u) \; | \; \underline{d} = \underline{D}, \; \widetilde{a} = \widetilde{A}\right\} \qquad (3-12)$$

be identically zero for $f(T_i) \le t,u \le T_f$. Using (3-7), we note that the *components of $\underline{v}(t,u)$ are*

$$v_1(t,u) = E\left\{[s(t) - \hat{s}_n(t)] \; [s(u) + n_1(u)]\right\} \qquad (3-13)$$

and

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$$v_2(t,u) = E\left\{ [s(t) - \hat{s}_n(t)] \, n_2(u) \right\} \tag{3-14}$$

for $f(T_i) \leq t, u \leq T_f$.

By the definitions of $f(t,u;\underline{D},\widetilde{A})$ and $g_n(t,u;\underline{D},\widetilde{A})$,

$$E\left\{[n_1(t) - \hat{n}_1(t)]n_2(u)\right\} = 0 \; ; \; f(T_i) \leq t, u \leq T_f \tag{3-15}$$

and

$$E\left\{[s(t) - \hat{s}_n(t)][s(u) + n_1(u) - \hat{n}_1(u)]\right\} = 0 \; ; \; f(T_i) \leq t, u \leq T_f. \tag{3.16}$$

Recall that the signal process $s(t)$ is orthogonal to the white noise processes $w_1(t)$ and $w_2(t)$. Therefore, $s(t)$ is also orthogonal to the noise processes $n_1(t)$ and $n_2(t)$ defined in (3-8).

It follows that (3-14) simplies to

$$v_2(t,u) = -E\left\{\hat{s}_n(t)n_2(u)\right\} \tag{3-17a}$$

which, with the aid of (3-10), becomes

$$v_2(t,u) = -E\left\{ \int_{f(T_i)}^{T_f} g_n(t,\sigma;\underline{D},\widetilde{A}) \, [s(\sigma) + n_1(\sigma) - \hat{n}_1(\sigma)]d\sigma]n_2(u) \right\}$$

$$= \int_{f(T_i)}^{T_f} g_n(t,\sigma;\underline{D},\widetilde{A})E\left\{[n_1(\sigma) - \hat{n}_1(\sigma)]n_2(u)\right\} d\sigma$$

$$= 0 \; ; \; f(T_i) \leq t, \, u \leq T_f , \tag{3-17b}$$

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where the last step follows from (3-15). The result that $v_2(t,u)$ in (3-14) equals zero and the fact that $\hat{n}_1(t)$ depends linearly upon $n_2(t)$ together imply

$$E\{[s(t) - \hat{s}_n(t)]\hat{n}_1(u)\} = 0 \; ; \; f(T_i) \leq t, \; u \leq T_f \; . \tag{3-18}$$

Substracting (3-18) from (3-13) leads to

$$v_1(t,u) = E\{[s(t) - \hat{s}_n(t)] \; [s(u) + n_1(u) - \hat{n}_1(u)]\}. \tag{3-19}$$

Comparing (3-19) with (3-16), we see that

$$v_1(t,u) = 0 \; ; \; f(T_i) \leq t, \; u \leq T_f \; . \tag{3-20}$$

This completes the proof.

The LMMSE estimator of $\tilde{a}s(t - d(t))$, $T_i \leq t \leq T_f$, from $\underline{r}(u)$, $T_i \leq u \leq T_f$, conditioned on $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, follows easily from the fact that $\tilde{a}s(t - d(t))$ is a linear transformation of $s(t)$. Because all available data have been used to obtain $\hat{s}_n(t)$, $f(T_i) \leq t \leq T_f$, the noncausal LMMSE estimate of $\tilde{a}s(t - d(t))$, given $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, is simply the scaled and delayed version of $\hat{s}_n(t)$ in (3-10), namely, $\tilde{A}\hat{s}_n(t - D(t))$.

The explicit form for $f(t,u;\underline{D},\tilde{A})$ follows easily from (3-9) and (3-15), which together imply

$$E\left\{n_1(t)n_2(u)\right\} = \int_{f(T_i)}^{T_f} f(t,\sigma;D,\tilde{A})E\left\{n_2(\sigma)n_2(u)\right\} d\sigma \ , \ f(T_i) \leq t, \ u \leq T_f \ .$$

$$(3\text{-}21)$$

This can be simplified by using (3-8) and (1-5), which imply

$$E\left\{n_2(t)n_2(u)\right\} = \begin{cases} \dfrac{N_o}{8}\left[\delta(t-u) + \tilde{A}^{-2}\ \delta(\beta(t) - \beta(u))\right]; \ T_i < u \leq f(T_f) \\[2ex] 0 \ ; \ \text{otherwise} \end{cases}$$

$$(3\text{-}22)$$

and

$$E\left\{n_1(t)n_2(u)\right\} = \begin{cases} \dfrac{N_o}{8}\left[\delta(t-u) - \tilde{A}^{-2}\ \delta(\beta(t) - \beta(u))\right]; \ T_i \leq u \leq f(T_f) \ . \\[2ex] 0; \ \text{otherwise} \end{cases}$$

$$(3\text{-}23)$$

Since $\beta(t)$ is a one-to-one function, then

$$\delta(\beta(t) - \beta(u)) = \frac{1}{|\dot{\beta}(u)|}\ \delta(t-u) \ ,$$

$$(3\text{-}24)$$

where the dot denotes the derivative of a function. It follows from (3-3) and (3-5) that

$$\dot{\beta}(u) = \frac{1}{1 - \dot{D}(\beta(u))} \ .$$

$$(3\text{-}25)$$

Note that $s(t - D(t))$ is locally reversed in time where $\dot{D}(t) > 1$ and frozen in time where $\dot{D}(t) = 1$. Therefore, it is reasonable to define $D(t)$ as a valid delay function if and only if

$$\dot{D}(t) \leq 1 \ ,$$

$$(3\text{-}26)$$

where the equality holds only at isolated values of t. It is easy to see that the above condition guarantees that $f(t)$ is invertible. By combining (3-24), (3-25), and (3-26), we obtain

$$\delta(\beta(t) - \beta(u)) = [1 - \dot{D}(\beta(u))] \ \delta(t - u) . \tag{3-27}$$

Therefore, (3-22) and (3-23) become, respectively,

$$E\{n_2(t)n_2(u)\} = \begin{cases} \dfrac{N_o}{8} \left[1 + \dfrac{1 - \dot{D}(\beta(u))}{\tilde{A}^2} \right] \delta(t - u) \ ; \ T_i < t \leq f(T_f) \\ \\ 0 \ ; \ \text{otherwise} \end{cases} \tag{3-28}$$

and

$$E\{n_1(t)n_2(u)\} = \begin{cases} \dfrac{N_o}{8} \left[1 - \dfrac{1 - \dot{D}(\beta(u))}{\tilde{A}^2} \right] \delta(t-u) \ ; \ T_i < u \leq f(T_f) . \\ \\ 0 \ ; \ \text{otherwise} \end{cases} \tag{3-29}$$

The substitution of (3-28) and (3-29) into (3-21) yields

$$f(t,u;\underline{D},\tilde{A}) = k(u) \ \delta(t - u) , \tag{3-30}$$

where

$$k(u) \ \triangleq \begin{cases} \dfrac{\tilde{A}^2 - [1 - \dot{D}(\beta(u))]}{\tilde{A}^2 + [1 - \dot{D}(\beta(u))]} \ ; \ T_i < u \leq f(T_f) . \\ \\ 0 \ ; \ \text{otherwise}. \end{cases} \tag{3-31}$$

Therefore,

$$\hat{n}_1(t) = k(t)n_2(t) . \tag{3-32}$$

Note that the MMSE estimate of $n_1(t)$ from $n_2(t)$ requires only multiplication of $n_2(t)$ by a time-varying gain.

It is interesting to observe from (3-29) that if

$$D(t) = D_0 + (1 - \widetilde{A}^2)t \tag{3-33}$$

then $n_1(t)$ and $n_2(t)$ become statistically independent and

$$\hat{n}_1(t) = 0 . \tag{3-34}$$

Equation (3-33) is a necessary and sufficient condition for (3-34). A sufficient condition arises when the delay is constant,

$$D(t) = D_0 , \tag{3-35a}$$

and the magnitude of the attenuation constant is unity,

$$\widetilde{A} = \pm 1 . \tag{3-35b}$$

These results are a consequence of the fact that the statistics of $w_2(t)$ are unchanged by the inverse operator (3-1) when $D(t)$ and $\widetilde{A}$ satisfy (3-34). The point is that if $d(t)$ and $\widetilde{A}$ are known a priori to satisfy (3-33) then the hypothetical quantities $D(t)$ and $\widetilde{A}$ can also be assumed to satisfy (3-3). This results in a simplified receiver because under these conditions $k(u)$ is identically zero.

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An equation specifying $g_n(t,u;\underline{D},\widetilde{A})$ can be obtained by using the fact that $g_n(t,u;\underline{D},\widetilde{A})$ is the LMMSE estimator of $s(t)$, $f(T_i) \leq t \leq T_f$, from

$$z(u) = s(u) + n(u) \; ; \; f(T_i) \leq u \leq T_f \; , \tag{3-36}$$

where

$$n(u) \stackrel{\Delta}{=} n_1(u) - \hat{n}_1(u) \; ; \; f(T_i) \leq u \leq T_f \; . \tag{3-37}$$

The noise process $n(u)$ is zero mean and uncorrelated with the signal process $s(u)$. Its covariance function is

$$\begin{aligned}
E\{n(t)n(u)\} &= E\{[n_1(t) - \hat{n}_1(t)][n_1(u) - \hat{n}_1(u)]\} \\
&= E\{[n_1(t) - \hat{n}_1(t)]n_1(u)\} \\
&= E\{n_1(t)n_1(u)\} - k(t)E\{n_2(t)n_1(u)\} \; .
\end{aligned} \tag{3-38}$$

By a derivation similar to that leading to (3-28) one finds

$$E\{n_1(t)n_1(u)\} = \begin{cases} \dfrac{N_0}{2\widetilde{A}^2} [1 - \dot{D}(\beta(u))] \, \delta(t - u) \; ; \; f(T_i) \leq t \leq T_i \\[3mm] \dfrac{N_0}{8} \left[1 + \dfrac{1 - \dot{D}(\beta(u))}{\widetilde{A}^2} \right] \delta(t - u) \; ; \; T_i < t \leq f(T_f) \\[3mm] \dfrac{N_0}{2} \, \delta(t - u) \; ; \; f(T_f) < t \leq T_f \; . \end{cases} \tag{3-39}$$

Combining (3-38), (3-39), and (3-31) gives

$$E\{n(t)n(u)\} = Q(u)\delta(t - u) \; ; \; f(T_i) \leq u \leq T_f \; , \tag{3-40}$$

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where

$$
Q(u) = \begin{cases}
\dfrac{N_o}{2} [1 - \dot{D}(\beta(u))] \; ; \; f(T_i) \leq u \leq T_i \\[12pt]
\dfrac{N_o [1 - \dot{D}(\beta(u))]}{2\left[\tilde{A}^2 + [1 - \dot{D}(\beta(u))]\right]} \; ; \; T_i < u \leq f(T_f) \\[12pt]
\dfrac{N_o}{2} \; ; \; f(T_f) < u \leq T_f \; ,
\end{cases}
\tag{3-41}
$$

and it follows that $n(t)$ is nonstationary white noise. The equation specifying $g_n(t,u;\underline{D},\tilde{A})$ is now obtained by substituting

$$
\hat{s}_n(t) = \int_{f(T_i)}^{T_f} g_n(t,\sigma;\underline{D},\tilde{A}) z(\sigma) d\sigma \; ; \; f(T_i) \leq t \leq T_f
\tag{3-42}
$$

into the orthogonality condition ,

$$
E\left\{ [s(t) - \hat{s}_n(t)] \, z(u) \right\} = 0 \; ; \; f(T_i) \leq t,u \leq T_f .
\tag{3-43}
$$

This leads directly to

$$
R_s(t,u) = \int_{f(T_i)}^{T_f} g_n(t,\sigma;\underline{D},\tilde{A}) \, R_s(\sigma,u) \, d\sigma
$$

$$
+ Q(u)g_n(t,u;\underline{D},\tilde{A}) \; ; \; f(T_i) < t,u < T_f .
\tag{3-44}
$$

Note that the problem of finding $g_n(t,u;\underline{D},\tilde{A})$ is equivalent to the problem of deriving the noncausal LMMSE estimator of $s(t)$ from $s(t) + n(t)$, where the process $n(t)$ is nonstationary white noise uncorrelated with $s(t)$.

If $s(t)$ is a state representable process, then $\hat{s}_n(t)$ can be obtained using optimal linear smoothing (reference 22, chapter 5).

3.3 EXPLICIT FORM FOR ENTRIES IN $\underline{H}_n(t,u;\underline{D},\widetilde{A})$

We have now specified the structure of $\underline{H}_n(t,u;\underline{D},\widetilde{A})$. This structure is shown in figure 3-4, where the filter $g_n(t,u;\underline{D},\widetilde{A})$ is given by the solution to (3-44) and where $k(t)$ is given by (3-31). Using this structure, we now derive the explicit form for the individual entries $h_{ij}(t,u;\underline{D},\widetilde{A})$ in $\underline{H}_n(t,u;\underline{D},\widetilde{A})$. This can be done be noting that, by definition, the output of $\underline{H}_n(t,u;\underline{D},\widetilde{A})$ is (see figure 3-4)

$$x(t) = \int_{T_i}^{T_f} h_{11}(t,u;\underline{D},\widetilde{A})r_1(u)du + \int_{T_i}^{T_f} h_{12}(t,u;\underline{D},\widetilde{A})r_2(u)du \qquad (3\text{-}45)$$

$$y(t) = \int_{T_i}^{T_f} h_{21}(t,u;\underline{D},\widetilde{A})r_1(u)du + \int_{T_i}^{T_f} h_{22}(t,u;\underline{D},\widetilde{A})r_2(u)\,du \ . \qquad (3\text{-}46)$$

On the other hand, by tracing the signals through the system in figure 3-4, we can express $x(t)$ and $y(t)$ in terms of $g_n(t,u;\underline{D},\widetilde{A})$ and $k(u)$. To keep the notation simple, we will subsequently write $g_n(t,u;\underline{D},\widetilde{A})$ as $g_n(t,u)$. An examination of figure 3-4 with the aid of figure 3-2 and equation (3-6) yields, after a little labor,
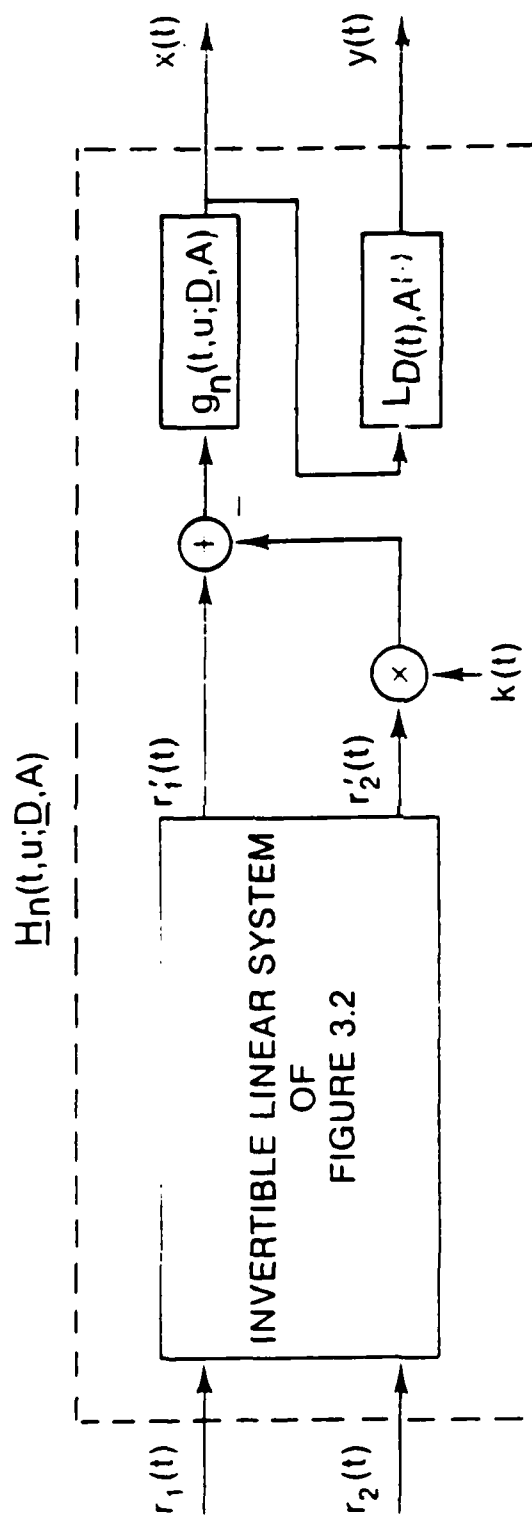
Figure 3-4. The System $\underline{H}_n(t,u;\underline{D},\tilde{A})$ (If $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, then outputs $x(t)$ and $y(t)$ are the noncausal LMMSE estimates of $s(t)$ and as($t - d(t)$), respectively.)

$$x(t) = \int_{f(T_i)}^{T_i} g_n(t,u) \frac{1}{\tilde{A}} r_2(\beta(u)) du$$

$$+ \int_{T_i}^{f(T_f)} g_n(t,u) \frac{1}{2} [1 - k(u)] r_1(u) du$$

$$+ \int_{T_i}^{f(T_f)} g_n(t,u) \frac{1}{2\tilde{A}} [1 + k(u)] r_2(\beta(u)) du$$

$$+ \int_{f(T_f)}^{T_f} g_n(t,u) r(u) du_1 \ . \tag{3-47}$$

By changing variables in the first and third integrals (set $\sigma = \beta(u)$) (3-47) becomes, after a little more labor,

$$x(t) = \int_{T_i}^{\beta(T_i)} g_n(t,\sigma - D(\sigma)) \frac{1}{\tilde{A}} r_2(\sigma) [1 - \dot{D}(\sigma)] d\sigma$$

$$+ \int_{T_i}^{f(T_f)} g_n(t,u) \frac{1}{2} [1 - k(u)] r_1(u) du$$

$$+ \int_{\beta(T_i)}^{T_f} g_n(t,\sigma - D(\sigma)) \frac{1}{2\tilde{A}} [1 + k(\sigma - D(\sigma))] r_2(\sigma) [1 - \dot{D}(\sigma)] d\sigma$$

$$+ \int_{f(T_f)}^{T_f} g_n(t,u) r_1(u) du \ . \tag{3-48}$$

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By comparing (3-48) with (3-45) we obtain for $T_i \leq t, u \leq T_f$

$$h_{11}(t,u;\underline{D},\widetilde{A}) = \begin{cases} g_n(t,u)\frac{1}{2}[1 - k(u)] \ ; \ T_i \leq u \leq f(T_f) \\ \\ g_n(t,u) \ ; \ f(T_f) < u \leq T_f \end{cases} \qquad (3-49)$$

$$h_{12}(t,u;\underline{D},\widetilde{A}) = \begin{cases} g_n(t,u - D(u))\frac{1}{\widetilde{A}}[1 - \dot{D}(u)] \ ; \ T_i \leq u \leq \beta(T_i) \\ \\ g_n(t,u - D(u)) \ \frac{1}{2\widetilde{A}}[1 + k(u - D(u))][1 - \dot{D}(u)] \ ; \ \beta(T_i) < u \leq T_f \ . \end{cases}$$

$$(3-50)$$

We note that since $\sigma - D(\sigma) = f(\sigma)$ then from (3-5) and (3-31)

$$k(\sigma - D(\sigma)) = \frac{\widetilde{A}^2 - [1 - \dot{D}(\sigma)]}{\widetilde{A}^2 + [1 - \dot{D}(\sigma)]} \ . \qquad (3-51)$$

The formulas for $h_{21}(t,u;\underline{D},\widetilde{A})$ and $h_{22}(t,u;\underline{D},\widetilde{A})$ in (3-46) can be obtained easily by noting from figure 3-4 that

$$y(t) = \widetilde{A}x(t - D(t)) \ , \qquad (3-52)$$

which leads to

$$h_{21}(t,u;\underline{D},\widetilde{A}) = \widetilde{A}h_{11}(t - D(t),u;\underline{D},\widetilde{A}) \qquad (3-53)$$

$$h_{22}(t,u;\underline{D},\widetilde{A}) = \widetilde{A}h_{12}(t - D(t),u;\underline{D},\widetilde{A}) \ , \qquad (3-54)$$

where $T_i \leq t,u \leq T_f$.

3.4 EXPLICIT FORM FOR BIAS l_B $(\underline{D}, \tilde{A})$

We can obtain the explicit form for the entries of the causal matrix impulse response $\underline{H}_c(t,u;\underline{D},\tilde{A})$ by noting that if $T_f = t$ then none of the data $\underline{r}(v)$, $T_i \le v \le T_f$, are future data. Thus, for $T_f = t$ and for $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, $\underline{H}_n(t,v;\underline{D},\tilde{A})$ becomes the impulse response $\underline{H}_c(t,v;\underline{D},\tilde{A})$ of the causal LMMSE estimate $\underline{\hat{S}}_c(t;\underline{D},\tilde{A})$ of $\underline{s}(t;\underline{d},\tilde{a})$ from $\underline{r}(v)$, $T_i \le v \le t$, and $g_n(t,\sigma)$ in (3-42), (3-43), and (3-44) becomes the causal LMMSE estimator $g_c(t,\sigma)$ of $s(t)$ from $z(\sigma)$, $f(T_i) \le \sigma \le t$. Therefore, we can obtain the components of $\underline{H}_c(t,v;\underline{D},\tilde{A})$ by replacing $g_n(t,u)$ in equations (3-49), (3-50), (3-53), and (3-54) with $g_c(t,u)$, where $g_c(t,u)$ is the solution to (3-44) for $T_f = t$, with $g_c(t,u) = 0$ for $t < u$. With $\underline{H}_c(t,v;\underline{D},\tilde{A})$ so determined, $\mathit{l}_B(\underline{D},\tilde{A})$ and $\underline{E}_c(t;\underline{D},\tilde{A})$ can be obtained directly from (2-36) and (2-37), respectively. For example, by straightforward substitution, (2-36) becomes

$$
\begin{aligned}
\mathit{l}_B(\underline{D},A) = &-\frac{1}{2}\int_{T_i}^{f(T_f)} g_c(\sigma,\sigma)\frac{1}{2}[1 - k(\sigma)]d\sigma \\
&-\frac{1}{2}\int_{f(T_f)}^{T_f} g_c(\sigma,\sigma)d\sigma \\
&-\frac{1}{2}\int_{T_i}^{\beta(T_i)} \tilde{A}g_c(\sigma - D(\sigma),\, \sigma - D(\sigma))\frac{1}{\tilde{A}}[1 - \dot{D}(\sigma)]d\sigma \\
&-\frac{1}{2}\int_{\beta(T_i)}^{T_f} \tilde{A}g_c(\sigma - D(\sigma),\, \sigma - D(\sigma))\frac{1}{2\tilde{A}}[1 + k(\sigma - D(\sigma))][1 - \dot{D}(\sigma)]d\sigma.
\end{aligned}
$$

$$(3\text{-}55)$$

This result can be put into a simpler form by changing the variable of integration in the last two integrals. With $\sigma - D(\sigma) \to \sigma$, (3-55) becomes

$$\ell_B(\underline{D},\widetilde{A}) = -\frac{1}{2} \int_{f(T_i)}^{T_f} g_c(\sigma,\sigma)d\sigma \ . \tag{3-56}$$

The minimum mean square error associated with $g_c(t,u)$ is

$$\xi_{oc}(t) = E\{(s(t) - \hat{s}_c(t))^2\}$$

$$= R_s(t,t) - \int_{f(T_i)}^{t} g_c(t,u)R_s(t,u)du$$

$$= Q(t)g_c(t,t) \ , \tag{3-57}$$

where the last step follows from (3-44), with $T_f = t$. An alternative expression for $\ell_B(\underline{D},\widetilde{A})$ can be obtained by substituting (3-57) into (3-56). This observation is important because if $s(t)$ is a state representable process, then $\xi_{oc}(t)$ can be obtained from the matrix Riccatti equation (reference 22, chapter 4.3).

4. CANONICAL REALIZATIONS

Our formulation of the delay estimation problems leads naturally to four canonical realizations, which are based upon well-known receiver structures for the detection of Gaussian signals in white Gaussian noise (reference 19, section 2.1). Here we simply point out the potential application of these structures in delay estimation. A more detailed development and comparison of these structures, with the view to obtaining practical estimation algorithms, appears to be a fertile area for future research.

The substitution of (2-23) into (2-19) (with $\underline{r}(.)$ replaced by $\underline{R}(.)$) yields

$$\ell_R(\underline{D}, \widetilde{A}) = \frac{1}{N_o} \int_{T_i}^{T_f} \underline{R}^T(t) \underline{\hat{s}}_n(t; \underline{D}, \widetilde{A}) dt \ . \tag{4-1}$$

The resulting ML estimator of $d(t)$ and $\widetilde{a}$ is shown in figure 4-1, where, following the terminology of Van Trees, it is referred to as Canonical Realization No. 1. Observe that this realization is a vector estimator-correlator analogous to the scalar estimator-correlator in figure 2-2 of reference 19. Here's how it works: The system tentatively hypothesizes that the unknown delay $d(t)$ is $D(t)$ and that the unknown attenuation $\widetilde{a}$ is $\widetilde{A}$, where $D(t)$ is a possible delay function and $\widetilde{A}$ is a possible relative attenuation constant. The received vector waveform $\underline{R}(t)$ is input to the noncausal conditional LMMSE estimator of $\underline{s}(t; \underline{d}, \widetilde{a})$, which is designed with
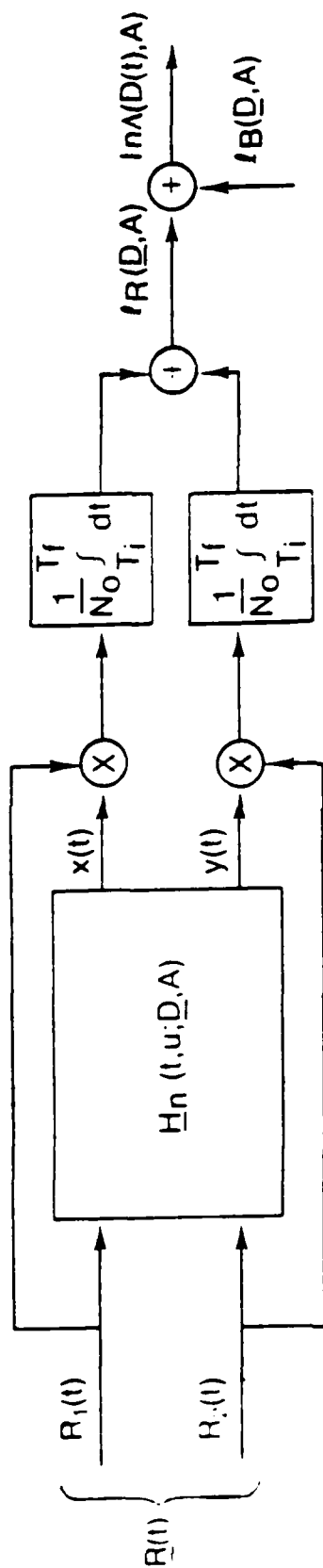
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Figure 4-1. Canonical Realization No. 1: Vector Estimator-Correlator Receiver
(When $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, then $x(t)$ and $y(t)$ are the noncausal LMMSE
estimates of $s(t)$ and $\tilde{a}s(t - d(t))$, respectively.)

the assumption that $\underline{D}$ and $\widetilde{A}$ represent the true values of delay vector $\underline{d}$ and attenuation scalar $\widetilde{a}$. The output of the estimator is $\hat{\underline{s}}_\eta(t;\underline{D},\widetilde{A})$ in (2-23). A possible realization of the estimator was shown in figure 3-4. The vector correlator then yields $\ell_R(\underline{D},\widetilde{A})$ in (4-1), which, when added to $\ell_B(\underline{D},\widetilde{A})$ of (2-36), (2-38), or (3-55), yields the value of the log-likelihood function $\ln\Lambda(D(t),\widetilde{A})$ for the assumed $D(t)$ and $\widetilde{A}$. This process is repeated for all choices of $D(t)$ and $\widetilde{A}$ that are possible for the application in question. The particular $D(t)$ and a jointly maximizing $\ln\Lambda(D(t),\widetilde{A})$ are the ML estimates $\hat{D}(t)]_{ML}$ and $\hat{\widetilde{A}}]_{ML}$ of $d(t)$ and $\widetilde{a}$.

An alternative form for Canonical Realization No. 1 can be obtained by noting that the lower integrator output in figure 4-1 can be written as

$$\frac{1}{N_0}\int_{T_i}^{T_f} y(t)\, r_2(t)\, dt = \frac{1}{N_0}\int_{T_i}^{T_f} \widetilde{A}\, x(t - D(t))\, r_2(t)\, dt$$

$$= \frac{1}{N_0}\int_{f(T_i)}^{f(T_f)} \widetilde{A}\, x(\sigma)\, r_2(\beta(\sigma))\, [1 - D(\beta(\sigma))]\, d\sigma\ .$$

$$(4-2)$$

The process $\widetilde{A}^{-1}\, r_2(\beta(u))$ is available at the output of the inverse operator $L_{\underline{D},\widetilde{A}}^{-1}\{r_2(u)\}$ in $\underline{H}_\eta(t,u;\underline{D},\widetilde{A})$, shown in figure 3-2, and $x(t)$ is the output of $g_\eta(t,u;\underline{D},\widetilde{A})$, shown in figure 3-4. Equation (4-2), therefore, provides a means for eliminating the operator $L_{D(t),\widetilde{A}}\{\cdot\}$ from the system in figure 3-4.

Alternative estimator structures can be obtained by straightforward

generalizations of the material in reference 19, pp. 15-23.

Canonical Realization No. 2, shown in figure 4-2, is a vector filter-correlator receiver. The matrix impulse response $\underline{H}'(t,u;\underline{D},\tilde{A})$ is defined by

$$\underline{H}'(t,u;\underline{D},\tilde{A}) = \begin{cases} \underline{H}_n(t,u;\underline{D},\tilde{A}) & ; \ t \geq u \\ 0 & ; \ t < u . \end{cases} \qquad (4\text{-}3)$$

Note that the output of the realizable filter $\underline{H}'(t,u;\underline{D},\tilde{A})$ is not the causal MMSE estimate of $\underline{s}(t;\underline{d},\tilde{a})$, given $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$.

Canonical Realization No. 3, shown in figures 4-3(a) and 4-3(b), are vector versions of the filter squarer receivers shown in figures 2-5 and 2-6 of reference 19. The matrix impulse responses $\underline{H}_{fn}(t,u;\underline{D},\tilde{A})$ and $\underline{H}_{fc}(t,u;\underline{D},\tilde{A})$ are the noncausal and causal solutions, respectively, to

$$\underline{H}_n(t,u;\underline{D},\tilde{A}) = \int_{T_i}^{T_f} \underline{H}_f(z,t;\underline{D},\tilde{A})\underline{H}_f(z,u;\underline{D},\tilde{A})dz, \ T_i \leq t, \ u \leq T_f . \qquad (4\text{-}4)$$

As with the scalar case, there are an infinite number of noncausal solutions because, with $\underline{H}_n(t,u;\underline{D},\tilde{A})$ given by (2-20),

$$\underline{H}_{fn}(t,u;\underline{D},\tilde{A}) = \sum_{i=1}^{\infty} \pm \frac{\lambda_i(\underline{D},\tilde{A})}{\lambda_i(\underline{D},\tilde{A}) + N_0/2} \ \underline{\phi}_i(t;\underline{D},\tilde{A})\underline{\phi}_i^T(u;\underline{D},\tilde{A}), \ T_i \leq t, \ u \leq T_f$$

$$(4\text{-}5)$$

is a solution to (4-4) for any assignment of plus and minus signs. The substitution of (4-4) into (2-19) yields

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Figure 4-2. Canonical Realization No. 2: Vector Filter-Correlator Receiver

Figure 4-3(a). Canonical Realization No. 3: Vector Filter-Crosscorrelator Receiver, Noncausal

Figure 4-3. Canonical Realization No. 3: Vector Filter-Crosscorrelator Receiver, Noncausal and Causal
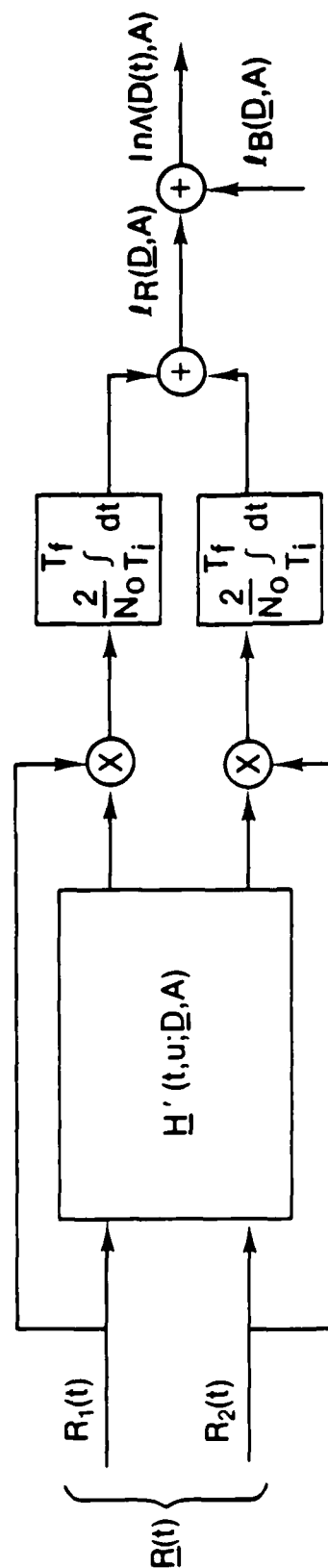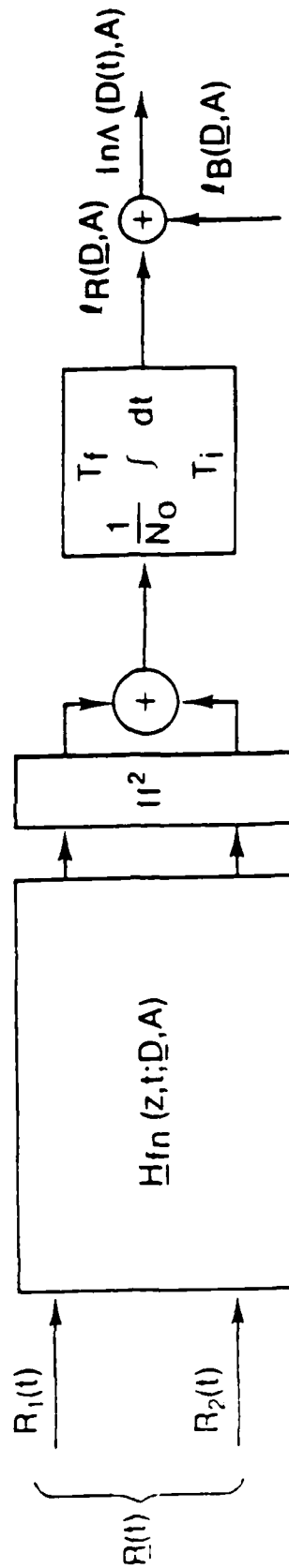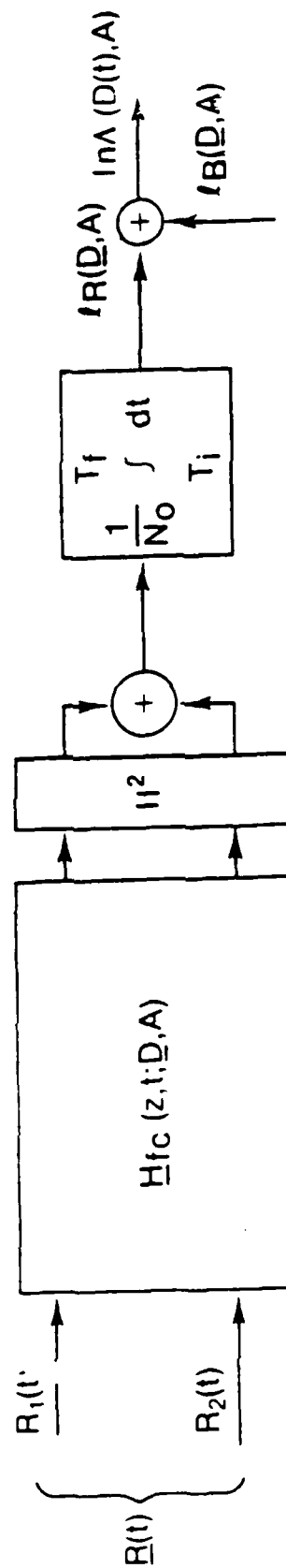
Figure 4-3(b). Canonical Realization No. 3: Vector Filter-Cross-correlator Receiver, Causal

Figure 4-3. Canonical Realization No. 3: Vector Filter-Crosscorrelator Receiver, Noncausal and Causal

$$\ell_R\ (\underline{D},\widetilde{A}) = \frac{1}{N_0} \int_{T_i}^{T_f} dz \left| \int_{T_i}^{T_f} \underline{H}_f(z,t;\underline{D},\widetilde{A})\underline{R}(t)dt \right|^2 .\tag{4-6}$$

A straightforward but lengthy generalization of the material in reference 3, pp. 19-24, leads to the expression

$$\ell_R\ (\underline{D},\widetilde{A}) = \frac{1}{N_0} \int_{T_i}^{T_f} \left\{ 2\underline{R}^T(t)\underline{\hat{s}}_c(t;\underline{D},\widetilde{A}) - \left| \underline{\hat{s}}_c(t;\underline{D},\widetilde{A}) \right|^2 \right\} dt ,\tag{4-7}$$

where $\underline{\hat{s}}_c(t;\underline{D},\widetilde{A})$ is the LMMSE causal estimate of $\underline{s}(t;\underline{d},\widetilde{a})$ from $\underline{R}(t)$, given $\underline{d} = \underline{D}$ and $\widetilde{a} = \widetilde{A}$. Equation (4.7) can be realized by the system shown in figure 4-4, which is referred to as Canonical Realization No. 4. The system $\underline{H}_c(t,u;\underline{D},\widetilde{A})$ in figure 4-4 is the matrix impulse of the casual LMMSE estimator encountered previously. Its structure can be obtained from $\underline{H}_n(t,u;\underline{D},\widetilde{A})$ by setting $T_f = t$. If $s(t)$ is state representable, then $g_c(t,u)$ can be realized using the Kalman filter (reference 22, chapter 4).
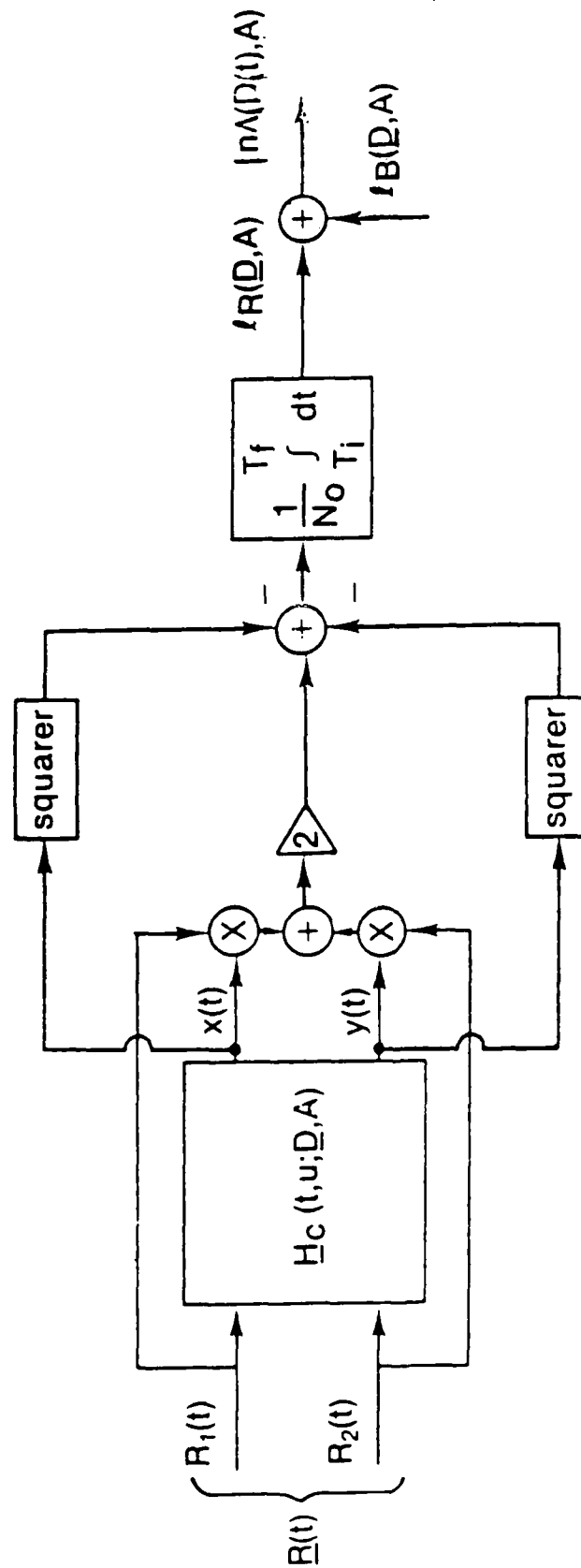
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Figure 4-4. Canonical Realization No. 4: Causal LMMSE Estimator Realization (When $\underline{d} = \underline{D}$ and $\tilde{a} = \tilde{A}$, then $x(t)$ and $y(t)$ are the causal LMMSE estimates of $s(t)$ and $\tilde{a}s(t - d(t))$, respectively.)

5.0 THE SPECIAL CASE OF CONSTANT DELAY, STATIONARY PROCESSES, AND LONG OBSERVATION INTERVAL (CDSPLOT)

This section links the general solution to the problem of ML time-delay estimation presented above to the solution of Knapp and Carter[1] in which the delay is constant, $d(t) = d_o$, the signal process is stationary, and the observation interval is long. We call this the CDSPLOT case. This exercise has the benefit of providing additional insight into Knapp and Carter's solution, as well as a more explicit description of the bias term, $\ell_B(D_o, \widetilde{A})$.

As was shown in (2-16) the log-likelihood function, $\ln \Lambda(\underline{D}, \widetilde{A})$, consists of the sum of a data-dependent term, $\ell_R(\underline{D}, \widetilde{A})$, and a bias term, $\ell_B(\underline{D}, \widetilde{A})$. The forms of these terms under the CDSPLOT approximation are derived in the next two subsections.

5.1 DATA—DEPENDENT TERM $\ell_R(D, \widetilde{A})$ UNDER CDSPLOT APPROXIMATION

The data-dependent term, $\ell_R(\underline{D}, \widetilde{A})$, is given in the general case in (2-19), where the entries of $\underline{H}_n(t, v; \underline{D}, \widetilde{A})$ are given in (3-49), (3-50), (3-53), and (3-54). It can be seen that the entries themselves are specified in terms of $g_n(t, u)$ and $k(u)$ in (3-44) and (3-31), respectively. Under the CDSPLOT approximation, we obtain $g_n(t, u)$ (approximately) by replacing T_i and T_f in (3-44) by $-\infty$ and $+\infty$, respectively, $R_s(t, u)$ by $R_s(t - u)$, and $Q(u)$ in (3.41) by

$$Q(u) = \frac{N_0'}{2} \qquad\qquad -\infty < u < \infty \ , \qquad\qquad (5\text{-}1a)$$

where

$$N_0' \triangleq \frac{N_0}{\widetilde{A}^2 + 1} \ . \qquad\qquad (5\text{-}1b)$$

(Equation (5-1) is obtained by setting $D(t) = D_0$ in (3-41) and noting that, with $[T_i, T_f] = [-\infty, \infty]$, the middle expression for $Q(u)$ in (3-41) applies for all $-\infty < u < \infty$.) With (3-44) so modified, we try a solution of the form $g_n(t, \sigma; \underline{D}, \widetilde{A}) = g_n(t - \sigma; D_0, \widetilde{A})$. Thus, under the CDSPLOT approximation, (3-44) becomes

$$R_s(t - u) = \int_{-\infty}^{+\infty} g_n(t - \sigma; D_0, \widetilde{A}) \, R_s(\sigma - u) \, d\sigma + \frac{N_0'}{2} g_n(t - u; D_0, \widetilde{A}) \qquad (5\text{-}2a)$$

for $-\infty < t, u < \infty$.

The above can be notationally simplified by a change of variables and by again assuming the dependence of $g_n(\cdot)$ on D_0 and $\widetilde{A}$ implicitly. This leads to

$$R_s(\tau) = \int_{-\infty}^{+\infty} g_n(\tau - \lambda) \, R_s(\lambda) \, d\lambda + \frac{N_0'}{2} g_n(\tau) \qquad\qquad (5\text{-}2b)$$

for $-\infty < \tau < \infty$.

The solution to (5-2b) can be obtained easily by Fourier transforms and is

$$g_n(t) = \int_{-\infty}^{+\infty} G(f) \, e^{j2\pi ft} df \, , \qquad (5-3)$$

where

$$G(f) = \frac{S_s(f)}{S_s(f) + N_0/2}, \qquad (5-4)$$

and where $S_s(f)$ is the power spectral density of $s(t)$. Since $s(t)$ is real, $S_s(f)$ and $G(f)$ are real and even. This implies that $g_n(t)$ is real and even:

$$g_n(-t) = g_n(t). \qquad (5-5)$$

It can also be seen from (5-2) that $g_n(t)$ does not depend on D_0. Also, with $D(t) = D_0$ and with $T_i = -\infty$, $T_f = +\infty$, $k(u)$ in (3-31) reduces to the constant

$$k(u) = \frac{\tilde{A}^2 - 1}{\tilde{A}^2 + 1} = k. \qquad (5-6)$$

The substitution of the above with $[T_i, T_f] = [-\infty, \infty]$, into (3-49), (3-50), and (3-53) will cause (3-54) to yield the time-invariant impulse responses,

$$h_{11}(t) = \frac{1}{\tilde{A}^2 + 1} \, g_n(t) \qquad (5-7)$$

$$h_{12}(t) = \frac{\tilde{A}}{\tilde{A}^2 + 1} \, g_n(t + D_0) \qquad (5-8)$$

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$$h_{21}(t) = \frac{\tilde{A}}{\tilde{A}^2 + 1} \, g_n(t - D_0) \qquad (5\text{-}9)$$

$$h_{22}(t) = \frac{\tilde{A}^2}{\tilde{A}^2 + 1} \, g_n(t) \; , \qquad (5\text{-}10)$$

which, when substituted into (2-19), yield

$$\ell_R \, (D_0, \tilde{A}) = \frac{1}{N_0} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \frac{1}{\tilde{A}^2 + 1} \, g_n(t - v) \, R_1(t) \, R_1(v) \, dt dv$$

$$+ \frac{1}{N_0} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \frac{\tilde{A}}{\tilde{A}^2 + 1} \, g_n(t - v + D_0) \, R_1(t) \, R_2(v) \, dt dv$$

$$+ \frac{1}{N_0} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \frac{\tilde{A}}{\tilde{A}^2 + 1} \, g_n(t - v - D_0) \, R_2(t) \, R_1(v) \, dt dv$$

$$+ \frac{1}{N_0} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} \frac{\tilde{A}^2}{\tilde{A}^2 + 1} \, g_n(t - v) \, R_1(t) \, R_2(v) \, dt dv \; . \qquad (5\text{-}11)$$

In (5-11) we have written the integration limits as $\pm\infty$ for convenience. Since $\underline{r}(t)$ is _defined_ as zero outside $[T_i, T_f]$, the integration is actually still over the long but finite interval $[T_i, T_f]$. Therefore, the integrals exist.

We are interested in finding the ML estimate of d_0, $\hat{D}_0]_{ML}$, which will be obtained by choosing D_0 to maximize $\ell_R(D_0, \tilde{A}) + \ell_B(D_0, \tilde{A})$. We will find in the next subsection that $\ell_B(D_0, \tilde{A})$ does not depend on D_0 under the COSPLOT assumption. Thus, it will be equivalent to maximize

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$\ell_R(D_0,\widetilde{A})$ in (5-11). Note that only the two middle terms in (5-11) depend on D_0. Using the fact that $g_n(t)$ is an even function, we note that these two terms can be combined and written as

$$\ell_R' (D_0,\widetilde{A}) = \frac{2}{N_0} \; \frac{\widetilde{A}}{\widetilde{A}^2 + 1} \iint\limits_{-\infty}^{+\infty} g_n(t - \sigma) \; R_1(t) \; R_2(\sigma + D_0) \; dtdv \; . \qquad (5-12)$$

Equation (5-12) can be written another way by introducing functions $h_1(t)$ and $h_2(t)$, satisfying the equation

$$\frac{2}{N_0} \; \frac{\widetilde{A}}{\widetilde{A}^2 + 1} \; g_n(t - \sigma) = \int_{-\infty}^{+\infty} h_1(z - t) \; h_2(z - \sigma) \; dz \; . \qquad (5-13)$$

The substitution of (5-13) into (5-12) yields

$$\ell_R' (D_0\widetilde{A}) = \int_{-\infty}^{+\infty} dz \int_{-\infty}^{+\infty} h_1(z - t) \; R_1(t) \; dt \int_{-\infty}^{+\infty} h_2(z - \sigma) \; R_2(\sigma + D_0) \; d\sigma \; , \qquad (5-14)$$

and we see that $\ell_R(D_0,\widetilde{A})$ can be obtained from the "generalized correlator" shown in reference 1, figure 1. By taking the Fourier transform in (5-13) it follows that

$$\frac{2}{N_0} \; \frac{\widetilde{A}}{\widetilde{A}^2 + 1} \; G(f) = H_1(f) \; H_2^\star(f) \equiv \Psi(f) \; , \qquad (5-15)$$

where $\Psi(f)$ is the "frequency weighting function" appearing in reference 1, equation (6). Combining (5-4) and (5-15) gives

$$\psi(f) = \frac{2}{N_o} \; \frac{\tilde{A}}{\tilde{A}^2 + 1} \; \frac{S_s(f)}{S_s(f) + N_o'/2} \; . \tag{5-16}$$

The function $\psi(f)$ can be written in terms of the coherence function of $r_1(t)$ and $r_2(t)$, defined as

$$\gamma_{12}(f) \triangleq \frac{S_{r_1 r_2}(f)}{\sqrt{S_{r_1}(f) \; S_{r_s}(f)}} \; , \tag{5-17}$$

where

$$S_{r_1}(f) = \text{power density spectrum of } r_1(t),$$

$$S_{r_1}(f) = S_s(f) + \frac{N_o}{2} \; , \tag{5-18}$$

$$S_{r_2}(f) = \text{power density spectrum of } r_2(t),$$

$$S_{r_2}(f) = \tilde{A}^2 \; S_s(f) + \frac{N_o}{2} \; , \tag{5-19}$$

$$S_{r_1 r_2}(f) = \text{cross-power-density spectrum of } r_1(t) \text{ and } r_2(t),$$

and

$$S_{r_1 r_2}(f) = \tilde{A} \; S_s(f) \; e^{+j2\pi f D_o} \; . \tag{5-20}$$

The detailed algebraic steps are shown below, which starts by substituting (5-2) into (5-16):

$$\psi(f) = \frac{\tilde{A}}{1 + \tilde{A}^2} \; \frac{2}{N_0} \; \frac{S_s(f)}{S_s(f) + \dfrac{N_0}{2(1 + \tilde{A}^2)}} \; .$$

$$= \frac{\tilde{A}S_s(f)}{(1 + \tilde{A}^2) \dfrac{N_0}{2} S_s(f) + \left(\dfrac{N_0}{2}\right)^2}$$

$$= \frac{\tilde{A}S_s(f)}{\left[S_s(f) + \dfrac{N_0}{2}\right]\left[\tilde{A}^2 S_s(f) + \dfrac{N_0}{2}\right] - \tilde{A}^2 S_s^2(f)}$$

$$= \frac{\left|S_{r_1 r_2}(f)\right| \; SGN(\tilde{A})}{S_{r_1}(f) \; S_{r_2}(f) - S_{r_1 r_2}^2(f)} \; . \tag{5-21}$$

By multiplying the numerator and denominator in (5-21) by

$$\left|S_{r_1 r_2}(f)\right| / [S_{r_1}(f) \; S_{r_2}(f)]$$

we obtain

$$\psi(f) = \frac{\left|\gamma_{12}(f)\right|^2 \; SGN(\tilde{A})}{\left|S_{r_1 r_2}(f)\right| \; [1 - \left|\gamma_{12}(f)\right|^2]} \; . \tag{5-22}$$

For $\tilde{A} > 0$, this is the frequency weighting function associated with the ML or HT (for Hannon/Thomson[2,3] processor in reference 1, table 1. Knapp and Carter[1] do not include the factor $SGN(\tilde{A})$ because their receiver has a square law device before the peak detector.

5.2 BIAS TERM $\ell_B(D_0, \widetilde{A})$ UNDER CDSPLOT APPROXIMATION

The bias term for the general case is given by (3-56), where $g_c(t,u)$ is the solution to (3-44) for $T_f = t$, with $g_c(t,u) = 0$ for $t < u$. Under the COSPLOT approximation, we set $D(t) = D_0$, $R_s(t,u) = R_s(t - u)$, and $T_f = t$ in (3-44) to obtain

$$R_s(t - u) = \int_{T_i - D_0}^t g_c(t,\sigma) \, R_s(\sigma - u) \, d\sigma + Q(u) \, g_c(t,u) \qquad (5\text{-}23)$$

for $T_i - D_0 < u \leq t$. Under the CDSPLOT approximation the function $Q(u)$ in (3-41) becomes

$$Q(u) = \begin{cases} \dfrac{N_0}{2} \; ; \; T_i - D_0 \leq u \leq T_i \\[2ex] \dfrac{N_0'}{2} \; ; \; T_i < u \leq t - D_0 \\[2ex] \dfrac{N_0}{2} \; ; \; t - D_0 < u \leq t \; , \end{cases} \qquad (5.24)$$

where N_0' was defined in (5-1b). Recall that $g_c(t,\sigma)$ is the impulse response of the casual LMMSE estimator of $s(t)$ from $z(t) = s(t) + n(t)$, where $n(t)$ has covariance function $Q(u)\delta(t - u)$ (see (3-36) through (3-40)). Looking at (5-24) we see that under the CDSPLOT approximation $n(u)$ is "piecewise" stationary in the three intervals $[T_i - D_0, T_i]$, $[T_i, t - D_0]$, and $[t - D_0, t]$; but $Q(u)$ changes abruptly at the interval boundaries. If we let $T_i \to -\infty$, then $n(u)$ will be stationary for $-\infty < u \leq t - D_0$ and $g_c(t,u)$ will be time-invariant in this range. Thus,

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$$g_c(t,u) = g_c^{(1)}(t - u) \quad \text{for } -\infty < u \le t - D_0 , \qquad (5\text{-}25)$$

where the function $g_c^{(1)}(v)$ is the solution to the Wiener-Hopf equation:

$$R_s(\tau) = \int_\infty^\infty g_c^{(1)}(v) \, R_z(\tau - v)dv \; ; \; 0 < \tau , \qquad (5\text{-}26)$$

with

$$R_z(\tau) = R_s(\tau) + \frac{N_0'}{2} \delta(\tau) \qquad (5\text{-}27)$$

and

$$g_c^{(1)}(\tau) = 0 \; ; \; \tau < 0 . \qquad (5\text{-}28)$$

Since the statistics of $n(u)$ change abruptly at $u = t - D_0$ and $g_c(t,u)$ operates, in general, over all past data, we cannot expect $g_c(t,u)$ to be time-invariant for $t - D_0 < u \le t$. Thus, under the COSPLOT approximation, $g_c(t,u)$ is a time-varying casual impulse response that has the approximate time-invariant form $g_c^{(1)}(t - u)$ specified in (5-26) for $T_i < u \le t - D_0$, but not otherwise. Using these results in (3-56), with $f(t) = t - D_0$, we have

$$\ell_B(D_0, \tilde{A}) \approx -\frac{1}{2} \int_{T_i - D_0}^{T_i} g_c(\sigma,\sigma)d\sigma - \frac{1}{2} \int_{T_i}^{t-D_0} g_c^{(1)}(0)d\sigma$$

$$-\frac{1}{2} \int_{t-D_0}^{t} g_c(\sigma,\sigma)d\sigma . \qquad (5\text{-}29)$$

Because D_0 is finite, the values of the first and the third integrals in (5-29) are negligible compared with that of the second integral for

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sufficiently large $t - T_i$. Consequently,

$$\ell_B (D_0, \tilde{A}) \simeq -\frac{1}{2} [t - D_0 - T_1] g_c^{(1)} (0)$$

$$\simeq -\frac{1}{2} [t - T_1] g_c^{(1)} (0) . \tag{5-30}$$

We can obtain a more explicit form for $\ell_B(D_0, \tilde{A})$ by referring to (3-57), which, under the COSPLOT approximation (for $T_i = -\infty < u < t - D_0$), becomes

$$\xi_{oc}(u) = E\{(s(u) - \hat{s}_c(u))^2\}$$

$$= \frac{N_0'}{2} g_c^{(1)}(0)$$

$$= \xi_{oc} . \tag{5-31}$$

If the signal spectrum $S_s(f)$ is rational with finite variance, then (reference 17, p. 501)

$$\xi_{oc} = \frac{N_0'}{2} \int_{-\infty}^{+\infty} \ln [1 + \frac{2}{N_0'} S_s(f)] df . \tag{5-32}$$

Combining (5-30), (5-31), and (5-32), we have

$$\ell_B (D_0, \tilde{A}) \simeq -\frac{1}{2} [t - T_i] \int_{-\infty}^{+\infty} \ln [1 + \frac{2}{N_0'} S_s(f)] df , \tag{5-33}$$

and we see that, as in reference 1, the bias does not depend upon D_0 for COSPLOT.

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6.0 SUMMARY

This report has generalized previous theory concerning ML time-delay estimation to include time-varying delay, finite observation interval, and nonstationary signal process. It has presented several receiver structures that can be used to obtain the ML estimates of time-delay and attenuation in one of two received signals compared with the other. Here it is shown that the general theory reduces to that of Knapp and Carter[1] for constant delay, stationary signal process, and long observation interval.

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